LONG WAVES ON WATER OF VARIABLE DEPTH

BY

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ABSTRACT

This dissertation is mainly a review of some of the work done by various authors on the long wave (shallow water) approximation and its applications to different problems. Shallow water wave equations are derived which are identical with Stoker's equations but the method of derivation is slightly different.

The method of characteristics is used in solving the differential equations governing the shallow water wave theory. The climbing and breaking of waves on sloping beaches is discussed. After the derivation of the transport equations for the discontinuities that can exist across a characteristic an equation is obtained for the time and hence the distance of breaking.

A simple explanation of the formation of a bore in a sloping stream is given here. The climb of a bore on a beach of uniform and non-uniform slope is also discussed briefly because of its close resemblance to non-uniform shock propagation in gas dynamics.

The notion of tsunami waves and the use of shallow water wave theory in the study of the numerical simulation of realistic tsunamis is also discussed briefly.

1. Introduction

The formation of waves in water is a natural phenomenon. Many mathematicians have formulated theories regarding various types of wave propagation. In Section 2 of this review a theory is developed to explain the behaviour of long waves in shallow water and although the method of derivation is slightly different the result is identical with the well established mathematical model first derived by Stoker[24], commonly known as the shallow water wave approximation which should more properly be called the long wave approximation. The theme of this dissertation is the long wave (shallow water) approximation and some of its different applications. A survey of some of the work done by various authors is included here.

The shallow water theory gives satisfactory results only in certain types of problems such as the climbing and breaking of waves over a sloping beach. Carrier and Greenspan (7) used Stoker's (24) non-linear shallow water theory to study breaking of waves and also the climbing of waves on a sloping beach without breaking. Their method was to specify the shape of the wave first and then to find the criteria for breaking, using the non-dimensionalised version of the shallow water equations and a constant slope beach.

In this dissertation we examine a similar problem using a more general method due to Jeffrey (12) which is directly applicable to problems in which the beach has a non-uniform slope.

A qualitative treatment of the problem of the change of form of waves moving into still water in the context of non-linear shallow water theory is discussed in Section 3. The method of characteristics which is the most convenient method for treating the initial value problems associated with the differential equations (2.15) and (2.16) is used in this work.

In Section 4 the propagation of waves having steadily decreasing and steadily increasing surface elevations are discussed. Then we proceed to discuss the continuity of the motion and prove that in the case of a wave having steadily increasing surface elevation the motion is continuous only up to the point of intersection of two characteristics which we interpret physically as the point of breaking of the wave.

The transport equations for the jump quantities concerned are derived in Section 5 and then we proceed to find the general criteria for breaking.

We are then inevitably led to consider the possibility of discontinuous motions. This type of motion, called a bore if the discontinuity front is a moving one and a hydraulic jump if it is stationary, is a common occurrence in nature.

In Section 6 the problem of the propagation of a smooth wave initially in the form of a sine wave into still water above a constant slope beach is treated. An equation is derived for the time, and hence the distance, of breaking which is defined as the point at which the slope of the front of the wave first becomes infinite.

The change in the form of pulses initiated as sine waves as they move into still water is illustrated by diagrams for a number of cases in Section 6.

In Section 7 we briefly discuss some of the specific problems treated by various authors. We also discuss briefly the notion of tsunami waves as treated by Mader (19) and the use of shallow water long wave theory in the study of the numerical simulation of realistic tsunami waves.

2. Formulation of the theory of Long Waves

We shall use the following equations of hydrodynamics and boundary conditions (24) in terms of Euler variables in deriving the shallow water theory.

Incompressibility

$$u'_{x'} + v'_{y'} + \omega'_{z'} = 0 \tag{2.1}$$

momentum equations

$$u'_{t'} + u'u'_{x'} + v'u'_{y'} + \omega'u'_{z'} = -\frac{p'_{x'}}{\rho} \tag{2.2a}$$

$$v'_{t'} + u'v'_{x'} + v'v'_{y'} + \omega'v'_{z'} = -\frac{p'_{y'}}{\rho} - g \tag{2.2b}$$

$$\omega'_{t'} + u'\omega'_{x'} + v'\omega'_{y'} + \omega'\omega'_{z'} = -\frac{p'_{z'}}{\rho} \tag{2.2c}$$

Irrotational condition

$$\omega'_{y'} = v'_{z'}, \quad u'_{z'} = \omega'_{x'}, \quad v'_{x'} = u'_{y'} \tag{2.2d}$$

Free surface condition

$$\eta'_{t'} + u'\eta'_{x'} + \omega'\eta'_{z'} = v' \quad \text{on} \quad y' = \eta' \tag{2.3}$$

$$p' = 0 \quad \text{on} \quad y' = \eta'$$

bottom condition

$$u'd'_{x'} - v' + \omega'd'_{z'} = 0 \quad \text{on} \quad y' = d'(x', z') \tag{2.4}$$

where u', v', ω' are the velocity components in the x', y', z' directions respectively.

p' is the pressure and subscripts denote derivatives in the directions indicated.

The co-ordinate axes are taken with the x', z' plane in the undisturbed water surface and the positive y' -axis upwards. The free surface elevation is given by $y' = \eta'(x', z', t')$ and the bottom surface by $y' = d'(x', z')$.

Cross-section by plane Z = 0

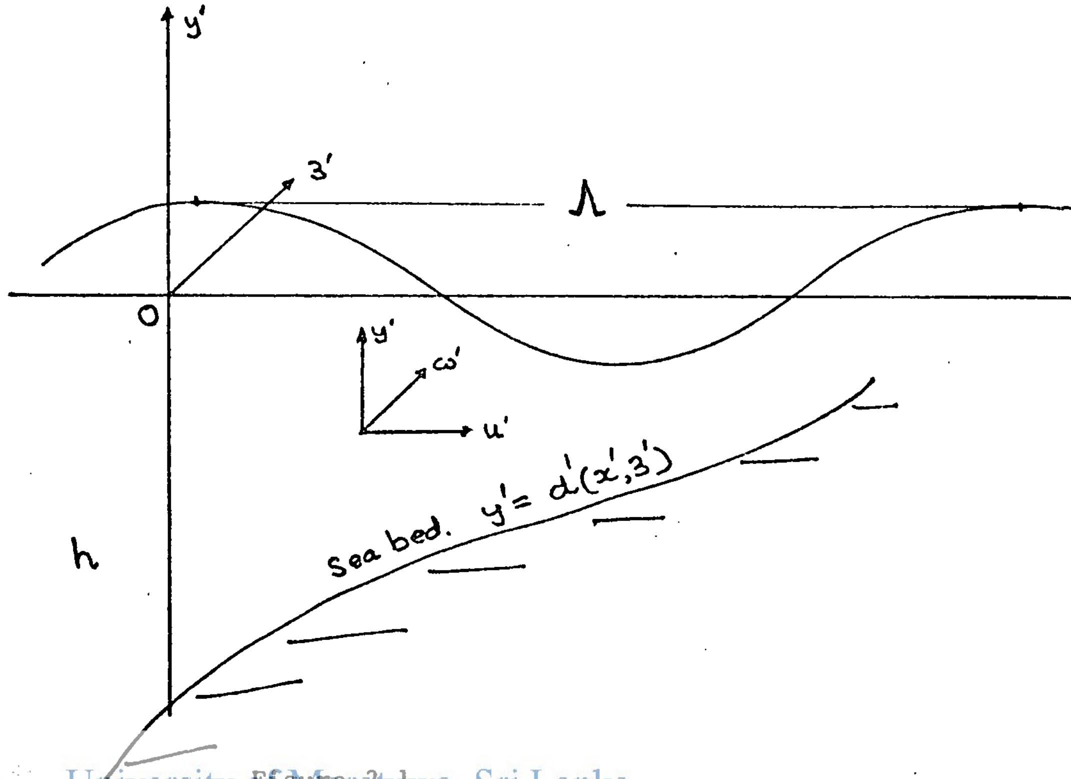


Figure 2.1

g = acceleration due to gravity

ρ = density of water (assumed to be constant)

h = undisturbed water depth at the origin of co-ordinates

Λ = characteristic wave length.

As already implied by (2.3), we assume the pressure to be zero on the free surface of water. In order to transform the equations (2.1) - (2.4) to equations with dimensionless variables we introduce the three quantities h, k, Λ with dimensions of length, and ε an arbitrary constant to be chosen. The transformation is as follows

$$x' = \Lambda x, \qquad u' = \varepsilon (gh)^{1/2} u.$$

$$y' = hy, \qquad v' = \varepsilon \frac{h}{\Lambda} (gh)^{1/2} v.$$

$$z' = kz, \qquad \omega' = \varepsilon \frac{k}{\Lambda} (gh)^{1/2} \omega.$$

$$t' = \frac{\Lambda}{(gh)^{1/2}} t, \quad \eta' = \varepsilon h \eta, \quad p' = \rho g h p, \quad d' = hH.$$

and $\quad c' = c c_o \quad$ where $\quad c_o = \sqrt{gh}.$

By substitution of these into (2.1) - (2.4) we get the following equations in terms of the new dimensionless variables

$$u_x + v_y + w_z = 0 \tag{2.5}$$

$$\varepsilon\{u_t + \varepsilon u u_x + \varepsilon v u_y + \varepsilon w u_z\} + p_x = 0 \tag{2.6a}$$

$$\varepsilon \frac{h^2}{\Lambda^2}\{v_t + \varepsilon u v_x + \varepsilon v v_y + \varepsilon w v_z\} + p_y + 1 = 0 \tag{2.6b}$$

$$\varepsilon \frac{k^2}{\Lambda^2}\{w_t + \varepsilon u w_x + \varepsilon v w_y + \varepsilon w w_z\} + p_z = 0 \tag{2.6c}$$

$$w_y = \frac{h^2}{k^2} v_z, \quad u_z = \frac{k^2}{\Lambda^2} w_x, \quad u_y = \frac{h^2}{\Lambda^2} v_x \tag{2.6d}$$

$$\eta_t + \varepsilon u \eta_x + \varepsilon w \eta_z = v \quad \text{on} \quad y = \varepsilon\eta \tag{2.7}$$

$$\begin{aligned} p &= 0 \quad \text{on} \quad y = \varepsilon\eta \\ uH_x + wH_z &= v \quad \text{on} \quad y = H \end{aligned} \tag{2.8}$$

Now we shall consider the effect of the ratio of the scale lengths in solving the equations (2.5) - (2.8). When the depth of the undisturbed water is small compared with the wave length, this ratio will be small. In solving the equations (2.5) and (2.6) we choose $\varepsilon = 1$, $\frac{k}{\Lambda} = 1$ and let $\frac{h}{\Lambda} \to 0$. It then follows from (2.6d) that u_x and w_z are, to first order independent of y, so that from (2.5) we have

$$v = -(u_x + w_z)y + \phi(x,z,t)$$

where ϕ is an arbitrary function of x, z and t.

Substituting for v in (2.8) we get

$$uH_x + wH_z = -(u_x + w_z)H + \phi(x,z,t)$$

that is

$$(uH)_x + (\omega H)_z = \Phi(x,z,t),$$

and so

$$v = -(u_x + \omega_z)y + (uH)_x + (\omega H)_z. \qquad (2.9)$$

Again from (2.6b) we have,

$$p_y + 1 = 0$$

Integrating with respect to y gives

$$p + y = F(x,z,t). \qquad (2.10)$$

where F is an arbitrary function of x, z and t.

But $p = 0$ on $y = \eta$

so that

$$\eta = F(x,z,t)$$

Thus

$$p(x,y,z,t) = \eta(x,z,t) - y \qquad (2.11)$$

Using (2.11) and (2.9) in (2.6a), (2.6c) and (2.7) together with the assumption that $v = 0$, we get

$$u_t + uu_x + \omega u_z + \eta_x = 0 \qquad (2.12)$$

$$\omega_t + u\omega_x + \omega\omega_z + \eta_z = 0 \qquad (2.13)$$

and

$$\eta_t + u\eta_x + \omega\eta_z = -(u_x + \omega_z)\eta + (uH)_x + (\omega H)_z.$$

that is

$$\eta_t + [u(\eta - H)]_x + [\omega(\eta - H)]_z = 0 \qquad (2.14)$$

(2.12), (2.13) and (2.14) are the non-linear shallow water equations for 3-dimensional flow over an arbitrary smooth bottom. Thus we arrive at the required shallow water theory. Some authors have chosen to use linear shallow water theory as the basis for the tidal wave theory in deep oceans. (Naturally oceans are not shallow, but their depth is very small compared with the wave length of tsunami waves so that the shallow water theory is appropriate).

We should note here that the functions u, ω and η depend on x, z and t only. If ω is taken to be zero, and we assume that all quantities are independent of z, we get the following equations

$$u_t + u u_x + \eta_x = 0 \tag{2.15}$$

$$\eta_t + \left[u(\eta - H) \right]_x = 0 \tag{2.16}$$

These differential equations are identical with the basic equations given by Stoker (24) except for the fact that the factor g, the acceleration due to gravity, is missing in (2.15) as we have introduced a dimensionless pressure.

3. Rearrangement of basic equations and proof of the existence of characteristics

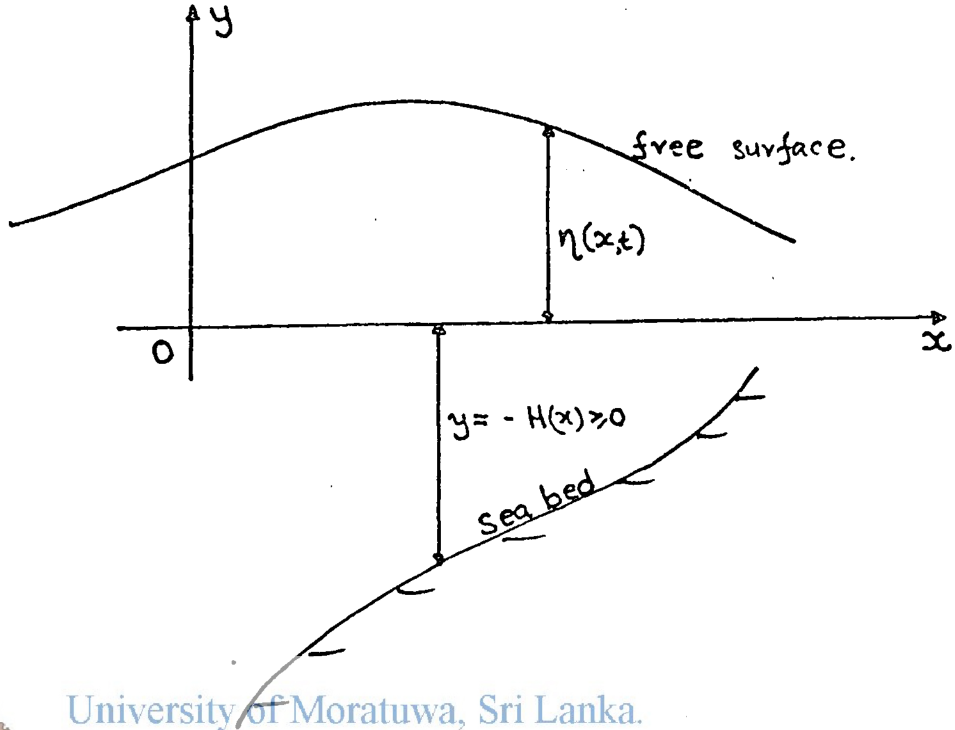


Figure 3.1

We shall commence this section with some definitions.

Definitions

Quasilinear systems

In a system of partial differential equations, if the elements of the unknown vector and its derivatives do not occur linearly, then the system is said to be non-linear. In a non-linear system, if the highest order derivatives of the unknown vector occur linearly then the system is said to be quasilinear. In its general form a first order quasilinear system may be written as

$$PU_t + QU_x + R = 0 \tag{3.1}$$

where P and Q are $n \times n$ matrices, P nonsingular, and R and U are $n \times 1$ column vectors. P, Q and R are all functions of U, x and t.

Instead of the general system (3.1), we may, without loss of generality, write

$$U_t + A\, U_x + B = 0 \tag{3.2}$$

where $A = A(U, x, t)$ is a $n \times n$ matrix, $B = B(U, x, t)$ is a $n \times 1$ column vector and $U = (u_1, u_2, \ldots, u_n)^T$ where $u_i = u_i(x,t)$, $i = 1, 2, \ldots n$ and T denotes the transpose operation. Unless otherwise stated the independent variables x and t are considered as scalars.

Weak and strong discontinuities

A discontinuity in the derivative of a function is called a weak discontinuity and a discontinuity in the function itself is called a strong discontinuity.

In this section we shall be considering the propagation of waves into water at rest above a sloping sea bed given by $y + H(x) = 0$. It follows from the previous theory that the surface wave velocity $c(x,t) = (H + \eta)^{1/2}$, η being the elevation of the free surface. Incorporating c in (2.15) and (2.16) we get

$$u_t + u u_x + 2 c c_x - H_x = 0 \tag{3.3}$$

and

$$2c_t + 2u c_x + c u_x = 0 \tag{3.4}$$

If the sea bed is of constant slope m, then $H_x = -m$ and the shallow water equations can be expressed in matrix form as given by Jeffrey (12) as follows,

$$U_t + A U_x + B = 0 \tag{3.5}$$

with t, x denoting the partial differentiation with respect to time and distance respectively and

$$U = \begin{bmatrix} u \\ c \end{bmatrix}, \quad A = \begin{bmatrix} u & 2c \\ \frac{c}{2} & u \end{bmatrix}, \quad B = \begin{bmatrix} m \\ o \end{bmatrix}$$

Our next problem is to find if the system (3.3) is hyperbolic. In order to achieve this we shall first change the variables from x, t to ϕ, t'', where

$$\phi = \phi(x, t'') \text{ and } t = t'' \tag{3.6}$$

It is well known that discontinuities of a derivative of a solution cannot occur except along characteristics (Courant and Lax (8)) and we assume that $\phi = 0$ to be one of the family of $C^{(u)}$ characteristics (to be defined later). Thus it is quite appropriate to take $\phi = 0$ as one of the new co-ordinate axes.

The new system of co-ordinates will be semi-curvilinear and for the transformation we shall use the following differential operators

$$\frac{\partial}{\partial t} \equiv \frac{\partial \phi}{\partial t} \cdot \frac{\partial}{\partial \phi} + \frac{\partial t''}{\partial t} \cdot \frac{\partial}{\partial t''} \tag{3.7}$$

and

$$\frac{\partial}{\partial x} \equiv \frac{\partial \phi}{\partial x} \cdot \frac{\partial}{\partial \phi} + \frac{\partial t''}{\partial x} \cdot \frac{\partial}{\partial t''} \tag{3.8}$$

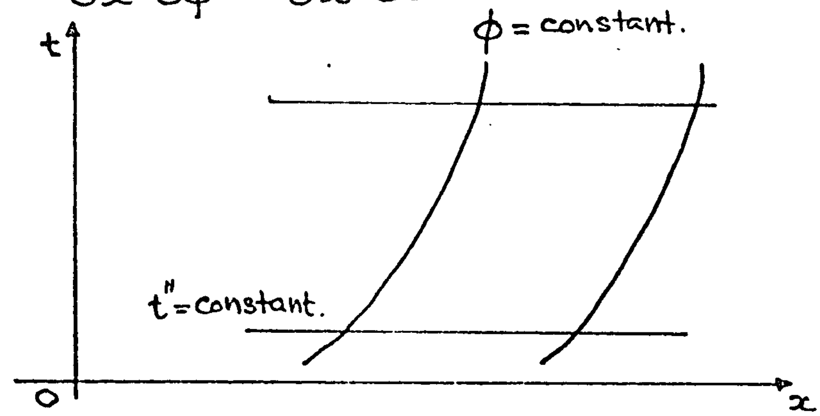


Figure 3.2

Since $t'' = t$, the operators become,

$$\frac{\partial}{\partial t} = \phi_t \frac{\partial}{\partial \phi} + \frac{\partial}{\partial t''}$$

and

$$\frac{\partial}{\partial x} = \phi_x \frac{\partial}{\partial \phi}$$

Thus the system (3.5) can be written in terms of the new variables as:

$$\phi_t U_\phi + U_{t''} + A \phi_x U_\phi + B = 0$$

that is, as

$$U_{t''} + (\phi_t + A\phi_x) U_\phi + B = 0 \qquad (3.9)$$

Now, along $\phi(x,t)$ = constant

$$\phi_x dx + \phi_t dt = 0$$

and so,

$$\frac{dx}{dt} = -\frac{\phi_t}{\phi_x} = \lambda, \qquad \text{say}$$

Hence (3.7) becomes

$$U_{t''}/\phi_x + (A - \lambda I) U_\phi + B/\phi_x = 0$$

Thus we can solve for $U_{t''}$, in terms of U_ϕ provided $|A - \lambda I| \neq 0$

The curves C defined by

$$C: \quad \frac{dx}{dt} = \lambda, \text{ where } |A - \lambda I| = 0,$$

are called characteristics if the eigen values λ are real and distinct. When the curves C are real the equations are then called hyperbolic. The eigen values of A in (3.5) are $\lambda^{(1)} = u + c$ and $\lambda^{(2)} = u - c$, so that they are real, and hence system (3.5) is hyperbolic with two sets of characteristic curves $C^{(1)}$ and $C^{(2)}$ corresponding, respectively, to $\lambda^{(1)}$ and $\lambda^{(2)}$

It has been established by many authors that discontinuities in the derivative of a solution of a quasi-linear hyperbolic system such as (3.5) are propagated along characteristics. Now we shall

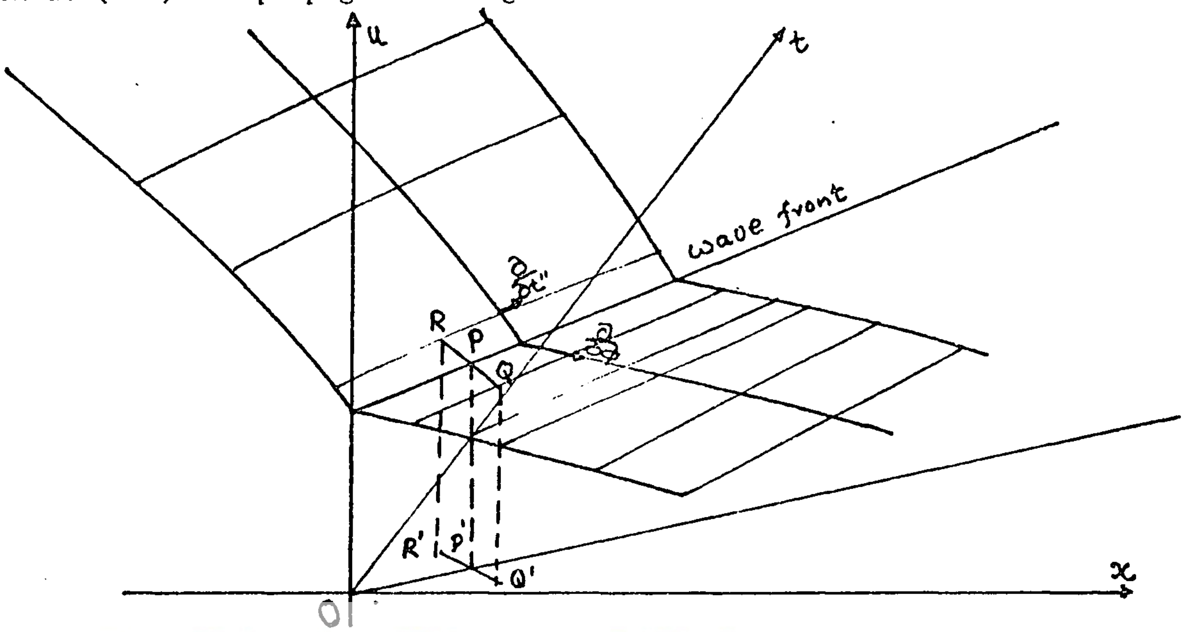


Figure 3.3

difference (3.9) across the wave front trace whose equation we take to be given by $\phi(x,t) = 0$. That is, we take $\phi = 0$ to bound the disturbed and undisturbed regions of water so that it represents the projection or trace in the (x,t) plane of the wave front. The parameterisation of the curves $\phi = \text{const}$, to achieve this will be discussed later. We know that $U_{t''}$ is continuous across $\phi(x,t) = 0$ and that U_ϕ is discontinuous. Hence if Q, R are points to the right and to the left of P on the wave front we can write,

$$(U_{t''})_R - (U_{t''})_Q + \left[(\phi_t I + A\phi_x)U_\phi\right]_R - \left[(\phi_t I + A\phi_x)U_\phi\right]_Q$$
$$+ (B)_R - (B)_Q = 0$$

or

$$\{\{U_{t''}\}\} + \{\{(\phi_t I + A\phi_x)U_\phi\}\} + \{\{B\}\} = 0 \qquad (3.10)$$

where
$$\{\{\cdot\}\} \equiv (\cdot)_R - (\cdot)_Q$$

If $Q, R \to P$, then the equation (3.10) becomes,

$$\left(\phi_t I + A\phi_x\right)_P \left[U_\phi\right] = 0 \qquad (3.11)$$

where,

$$\left[U_\phi\right] = \left[U_\phi\right]_{P-} - \left[U_\phi\right]_{P+}$$

Since we have taken P arbitrarily on the wave front we can drop the suffix P and can write (3.11) as,

$$\left(\phi_t I + A\phi_x\right)\left[U_\phi\right] = 0 \qquad (3.12)$$

The wave front trace is,

$$\phi(x,t) = 0$$

so, as we have already seen,

$$\phi_x dx + \phi_t dt = 0$$

whence,

$$\phi_x \frac{dx}{dt} + \phi_t = 0 \qquad (3.13)$$

Now from (3.10) and (3.11)

$$\left(A - \frac{dx}{dt} I\right) \phi_x \left[U_\phi\right] = 0 \qquad (3.14)$$

Setting
$$\lambda = \frac{dx}{dt} = -\frac{\phi_t}{\phi_x}, \qquad (3.13)'$$

equation (3.14) becomes

$$\left(A - \lambda I\right)\left[U_\phi\right] = 0, \qquad \text{provided } \phi \neq 0 \qquad (3.15)$$

There will be a non-trivial solution for U_ϕ if and, only if,

$$\left| A - \lambda\ I \right| = 0 \tag{3.16}$$

In general if A is an $n \times n$ matrix there will be n eigen values, $\lambda^{(1)}, \lambda^{(2)}, \ldots \lambda^{(n)}$ and hence n eigen vectors for the solution of (3.16). The curves ϕ = constant will be real if and only if all λ are real and distinct. Further, if all the eigen vectors are linearly independent, we call the system (3.5) totally hyperbolic. This again is our hyperbolicity condition appearing in the context of the derivative U_ϕ of U normal to the wave front trace.

In our problem the eigen values of A are given by,

$$\begin{vmatrix} u-\lambda & 2c \\ \frac{c}{2} & u-\lambda \end{vmatrix} = 0$$

or by,

$$\lambda^{(1)} = u + c \quad \text{and} \quad \lambda^{(2)} = u - c, \tag{3.17}$$

and the corresponding left eigen vectors $\ell^{(i)}$ are

$$\ell^{(1)} = [1, 2] \text{ and } \ell^{(2)} = [1, -2]. \tag{3.18}$$

Thus, as already stated, the system (3.5) has two families of characteristics $C^{(1)}$ and $C^{(2)}$ determined by the solutions of the equations

$$C^{(i)}: \quad \frac{dx}{dt} = \lambda^{(i)} \tag{3.19}$$

for $i = 1, 2$

The characteristics $C_0^{(1)}$ and $C_0^{(2)}$ originating from 0 in the (x, t) plane will be as shown in the figure 3.4

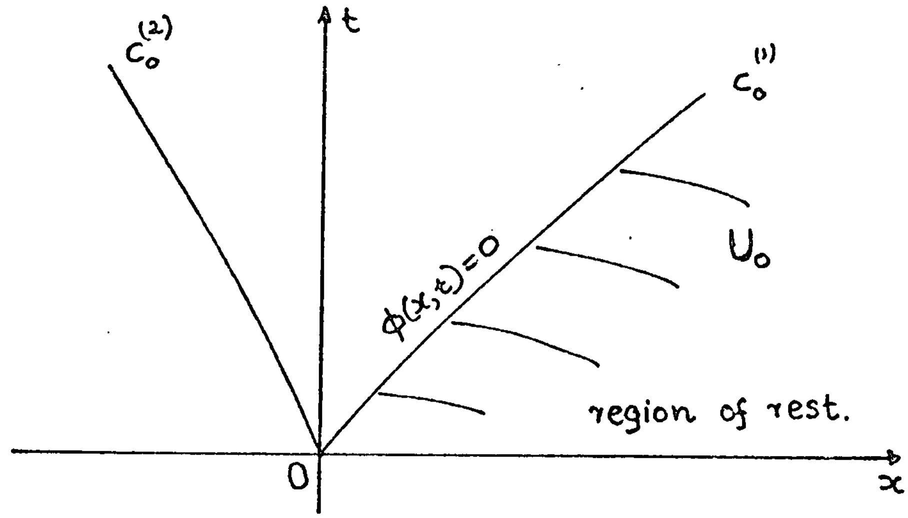


Figure 3.4

If the disturbance is going in to the right region which is at rest (for simplicity), then

$$u_o = 0$$

and

$$C_o(x) = \sqrt{1 - mx} \qquad (3.20)$$

From (3.17) and (3.19) we can determine the form of the characteristic $C_o^{(1)}$ (say) which forms the wave front through O as a function of time.

$$C_o^{(1)} : \frac{dx}{dt} = C_o$$

that is,

$$\frac{dx}{dt} = \sqrt{1 - mx} \qquad (3.21)$$

which when integrated gives,

$$x = t - \frac{mt^2}{4} \qquad (3.22)$$

Now premultiply the system (3.5) by $\ell^{(i)}$ and use the relation $\ell^{(i)}A = \lambda^{(i)}\ell^{(i)}$ to get,

$$\ell^{(i)}\left(U_t + \lambda^{(i)} U_x\right) + \ell^{(i)} B = 0 \tag{3.23}$$

along $C^{(i)}$.

Using the values of $\ell^{(i)}$, U and B, we get,

$$u_t \pm 2C_t + m = 0 \tag{3.24}$$

and on integrating (3.24) with respect to time, we find that

$$u + 2c + mt = \text{Constant (say } k_1) \text{ along } C^{(1)} \tag{3.25}$$

and

$$u - 2c + mt = \text{Constant (say } k_2) \text{ along } C^{(2)} \tag{3.26}$$

We shall assume that initially at t = 0, the values of u and c are prescribed, say by

$$\left.\begin{aligned} u(x,0) &= \bar{u}(x) \\ c(x,0) &= \bar{c}(x) \end{aligned}\right\} \tag{3.27}$$

By use of the initial conditions we can determine the constant K_1 and K_2. Then the values of u and c for all points in the (x,t) plane can be obtained by solving the differential equation (3.21) for the characteristic through that point. As ϕ is so far specified only by the differential equation (3.13), in which for waves moving to the right $\lambda = \lambda^{(1)} = u + c$, we must give initial conditions for ϕ as it is a co-ordinate variable for which $\phi = 0$ is the wave front through the origin. We set $\phi(x,0) = x$, when $\phi > 0$ ahead of the wave front and $\phi < 0$ behind it. As $1\Big/\left(\frac{\partial \phi}{\partial x}\right)$

is the Jacobian of (3.6), this implies that initially no two members of the same family of characteristic curves are tangent to each other, so that initially they will constitute a regular semi-curvilinear co-ordinate system in the (x,t) plane. We shall discuss later the case when the characteristics cease to have this property.

Simple Wave

It is possible to have a problem in which (a) the initial undisturbed depth of water is constant, (b) the water extends from the initial point to infinity at least in one direction and (c) the water is either at rest or moving with constant velocity with its free surface at zero elevation at the time t = 0. Under these conditions we see that one of the characteristic families of (3.21) consists of straight lines along which u and c are constants, and the corresponding motion is called a simple wave.

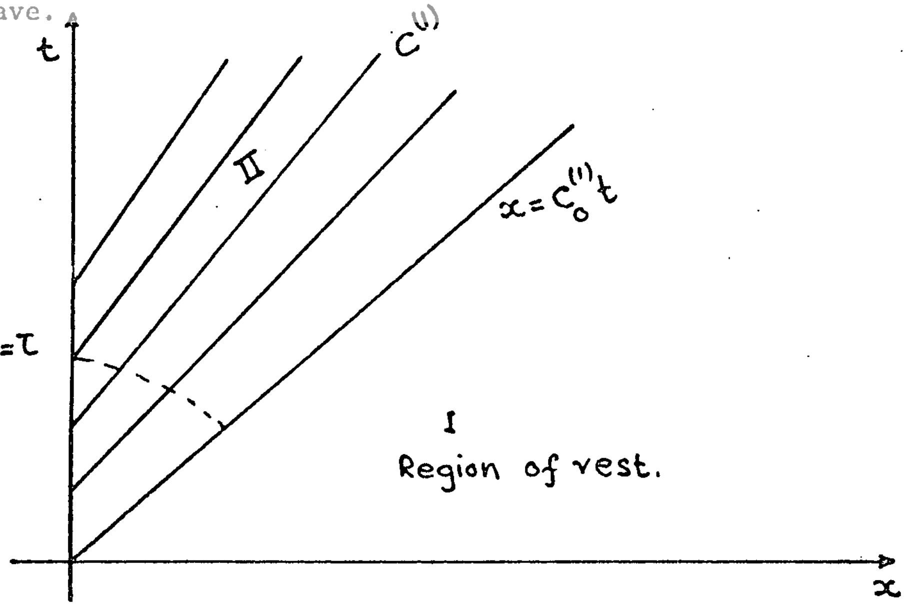


Figure 3.5

When a disturbance is initiated at t = 0, it is propagated into the region of rest and the water will remain at rest until the disturbance reaches that point. The nature of the motion is determined by the character of the disturbance at $x = 0$. Since there is a disturbance in the region II (see Fig. 3.5), one set of characteristics will be straight lines and the other curved lines. This has to be so, or else if both sets of characteristics are straight lines, then that region will be at rest.

4. Propagation of Waves into still water

There is a remarkable difference between the propagation of waves having a steadily decreasing surface elevation η at x = 0 and those having a steadily increasing surface elevation η at x = 0.

The slope of any characteristic issuing from t = τ on the t axis (Fig. 3.5) is given by,

$$\frac{dx}{dt} = 3\,C(\tau) - 2C_0 \qquad (4.1)$$

This equation gives us a complete family of straight characteristics. Now if $\eta(x,t)$ at x = 0 is a decreasing function, then $C(t)$ decreases with increase of time. Hence the slope $\frac{dx}{dt}$ of these straight line characteristics decreases as t increases and we get a family of straight characteristics which diverge on moving from the t axis. In the other case the function $C(t)$ increases as t increases and hence $\frac{dx}{dt}$ also increases as t increases. Thus the characteristics converge to a point. In the first case the motion is continuous throughout and in the second case the motion is continuous only up to the point of intersection of two characteristics. Physically we say that the wave breaks or developes a bore once the solution ceases to be continuous.

The propagation of waves into still water has been investigated by many authors. Jeffrey (1_ considered a smooth fronted wave, (that is a wave whose slope is continuous in the free surface, but which has a discontinuity in the derivative of the surface slope across some line in the free surface). The author also assumes an arbitrarily smooth sea bed profile and establishes the fact that breaking at the wave front cannot take place until the waves reach the shore line. Jeffrey and Tin (14) considered the propagation of non-smooth fronted waves over vertical walled objects on a flat sea bed. In a problem of this type, where the sea bed is not continuous and smooth, the reflection of waves is very significant. Carrier and Greenspan (7) considered a sea bed of constant slope and showed that breaking depends on the initial shape of the wave and the particle velocity distribution. There is still no general theory for the propagation of waves into water above a sea bed of arbitrary shape.

Waves in channels

In deriving the differential equations of the flow in open channels we should consider the existence of significant forces other than gravity, such as friction. Stoker (24) using the one dimensional theory obtained the following differential equations.

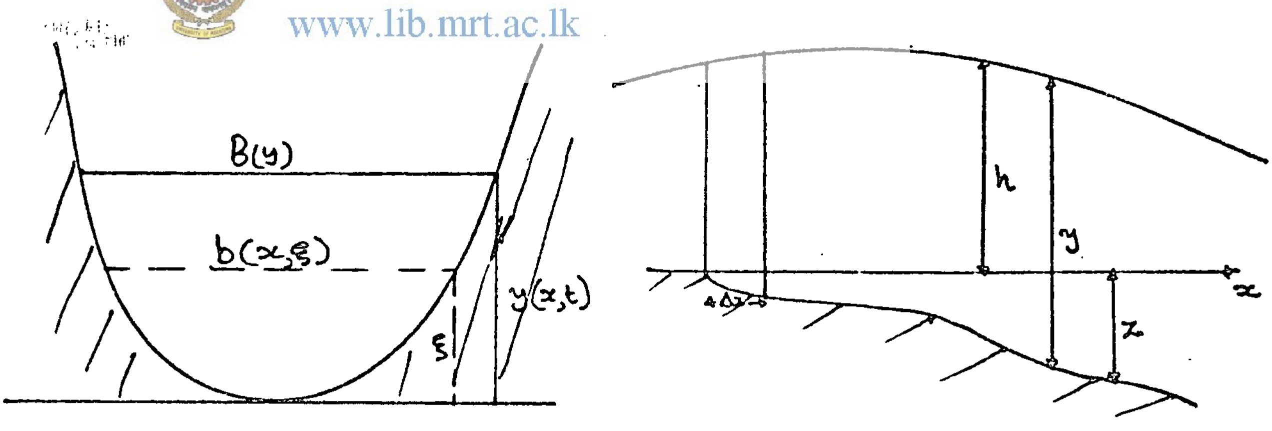


Figure 4.1

equation of continuity,

$$(Av)_x + A_t = q \tag{4.2}$$

equation of motion,

$$v_t + v v_x + \frac{q}{A} v = Sg - S_f g - y_x g \tag{4.3}$$

where A = area of cross-section

q = influx per unit length of channel

S = $\frac{dz}{dx}$, the slope

S_f = the friction slope

v = the velocity

and y = the depth

The differential equations governing the flow are expressions of the laws of conservation of mass and momentum. In deriving them the following assumptions are made. 1) The pressure in water obeys the hydraulic pressure law, 2) the slope of the bed of the river is small, 3) the effects of friction and turbulence can be represented by a resistance force depending on the square of the velocity v. These equations are used in describing certain types of waves that form in open channels.

Roll Waves

Sometimes in open channels there exists a flow in the form of a progressing wave moving downstream at constant speed without change in shape. This type of wave could be expressed mathematically as a function of depth $y(x,t)$ and velocity $v(x,t)$. Stoker (24) expressed y and v in the following form,

$$y(x,t) = y(x - Ut) \tag{4.4}$$

$$v(x,t) = v(x - Ut) \tag{4.5}$$

where U is a constant.

Introducing a new variable ξ where $\xi = x - Ut$, Stoker (24) transformed (4.2) and (4.3) to,

$$(v - U)y_\xi + y v_\xi = 0 \tag{4.6}$$

and

$$(v - U)v_\xi + g y_\xi + g(S_f - S) = 0 \tag{4.7}$$

for a rectangular channel of fixed breadth and slope. The solution of (4.6) and (4.7) is

$$\left(g - \frac{D^2}{y^3}\right)y_\xi + g(S_f - S) = 0,$$

where

D = constant of integration of (4.6).

With the aid of this differential equation it is possible to describe a special type of progressing wave in a uniform channel known as roll waves (Fig. 4.2).

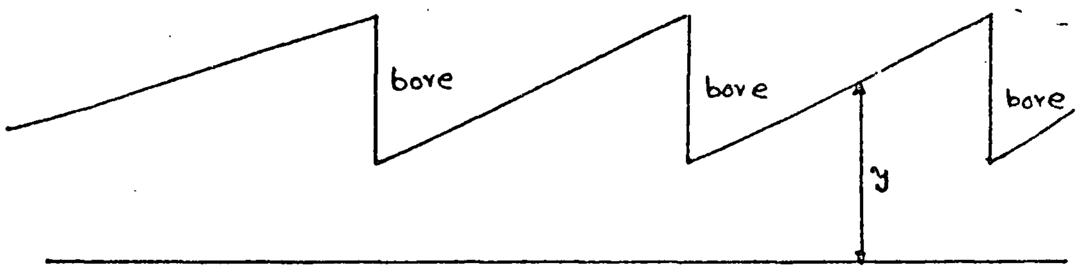


Figure 4.2

These waves consist of a series of bores connected by a stretch of smooth flow. This type of wave occurs in steep channels.

Practical observations lead one to wonder whether there are discontinuous periodic solutions with discontinuities in the form of bores. But the "shock" or bore conditions were derived assuming no resistance was present and it was confirmed by Dressler (8) that the resistance terms play no role in shock conditions. Dressler also proved that roll waves cannot occur either if the resistance is zero or if the resistance exceeds a certain critical value. As the resistance decreases, the size of the wave decreases and if the resistance becomes large the profiles reverse their directions and can no longer be joined by shocks. According to Dressler (8) the critical value is reached when the dimensionless resistance coefficient equals one-fourth the value of the channel slope.

Problems in non-viscous unsteady flow can be solved by using the Riemann method of characteristics to integrate the partial differential equations. The problem of roll-waves cannot be treated in this way as this type of flow depends basically upon resistance effects.

Solitary Waves

There exist waves of finite amplitude consisting of a single elevation which propagate without change of shape. Such types of wave are called solitary waves.

A theory was developed by Keller (16) by extending the theory of Friedrichs (9) to second order terms to get both solitary and enoidal type waves. In the lowest order approximation, the only possibility is the uniform flow with undeformed free surface. But if the speed U of the flow is taken to be critical, that is, $U = \sqrt{gh}$, where h is the undisturbed depth, then a bifurcation phenomenon takes place and the second order terms in the development of Friedrich's theory lead to the possibility of solitary and enoidal waves.

Stoker (24) has shown that a steady flow with critical speed $U = \sqrt{gh}$ is highly unstable, since the slightest disturbance leads to a motion where infinite elevations of the free surface occur in the context of the linear theory. Thus we should adopt the non-linear theory to explain solitary waves.

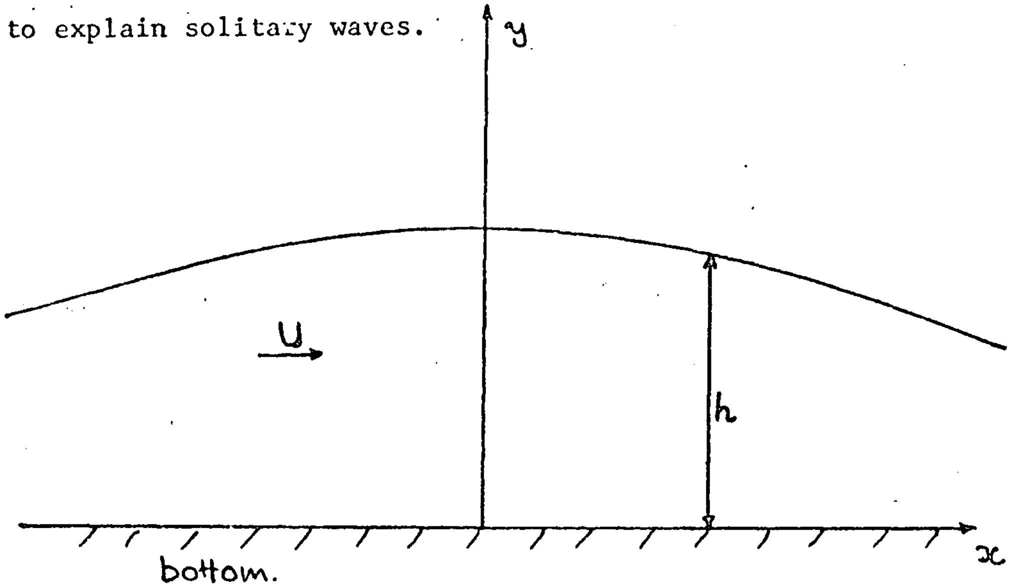


Figure 4.3

Solitary Wave

Stoker obtained the shape of the wave as,

$$y = 1 + 3\chi^2 \operatorname{Sech}^2 \frac{3\chi x}{2},$$

when the horizontal component of the velocity u is

$$u = 1 - 3\chi^2 \operatorname{sech}^2 \frac{3\chi x}{2},$$

and where χ is a quantity which depends on the speed U . These equations show that the solitary wave is of symmetrical form and the amplitude increases with the increase of speed U . A review of the occurrence of solitary waves and of their properties is to be found in the work of Jeffrey and Kakutani (15).

5. Transport Equations

We shall use the method due to Jeffrey (12) in deriving the transport equations for the discontinuities that can exist across $C_0^{(1)}$ in the case of a wave advancing up a sloping beach. Hereafter we call these C^1 discontinuities.

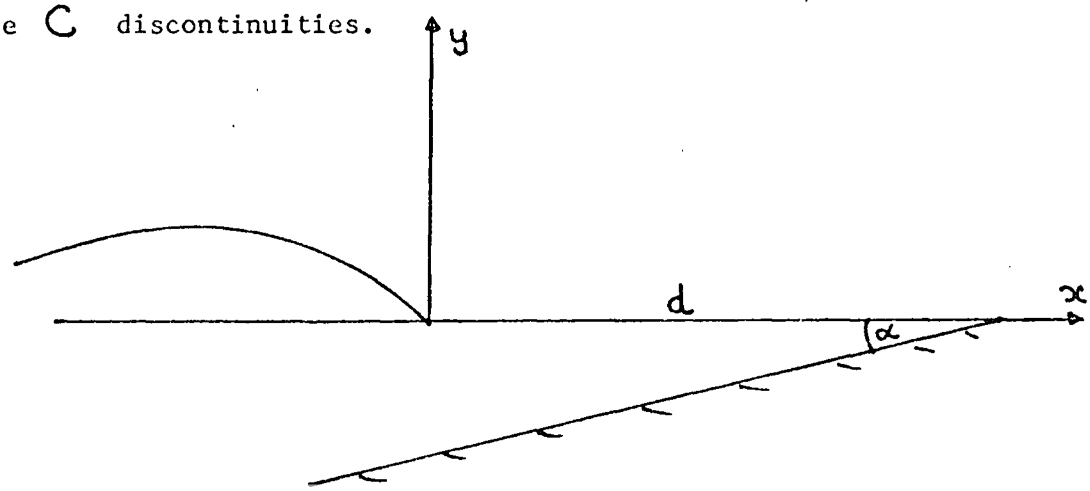


Figure 5.1

Assume that at $t = 0$, a periodic wave exists in the region $x \leq 0$. (Fig. 5.1) and that the water is undisturbed in the region $0 < x \leq d$; $x = d$ is the shore line. Let the disturbance η be given by

$$\eta(0,t) = A \sin \omega t. \tag{5.1}$$

The initial values suffer a Lipschitz discontinuity across the wave front. These discontinuities are propagated along the $C_0^{(1)}$ characteristic originating at the origin. We say that (mathematically) the breaking of wave at the wave front will take place first at (x_c, t_c) in the (x,t) plane when the $C^{(1)}$ characteristics intersect on the $C_0^{(1)}$ characteristic to form a cusp and the slope of the wave front at this point becomes infinite.

Now let us introduce the new variables

$$\phi(x,t) = \text{constant and } t'' = t \tag{5.2}$$

and as shown earlier require ϕ to satisfy the equation

$$\phi_t + \lambda^{(1)}\phi_x = 0 \tag{5.3}$$

where $\frac{dx}{dt} = \lambda^{(1)}$ along ϕ = constant.

We shall also impose the initial condition

$$\phi(x,0) = x \tag{5.4}$$

to obtain the parameterisation for ϕ described previously.

The wave front is then given by $\phi(x,t) = 0$ and ahead of the wave front $\phi(x,t) > 0$. The transformation (5.2) is non-singular if $x_\phi = \frac{1}{\phi_x}$ is non-zero. From the initial conditions, $x_\phi = 1$, and hence the Jacobian is non zero. We seek to find the condition that the Jacobian is zero on $C_0^{(1)}$. In terms of the new variables ϕ, t'' we have

$$\left.\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial\phi}{\partial t}\cdot\frac{\partial}{\partial\phi} + \frac{\partial}{\partial t''} \\ \text{and} \quad \frac{\partial}{\partial x} &= \frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial\phi}\cdot \end{aligned}\right\} \tag{5.5}$$

From (3.21), (5.3) and (5.5),

$$\ell^{(i)}\left\{\phi_t U_\phi + U_{t''} + \lambda^{(i)}\phi_x U_\phi\right\} + \ell^{(i)}B = 0$$

or

$$\ell^{(i)}\left\{U_{t''} + (\phi_t + \lambda^{(i)}\phi_x)U_\phi\right\} + \ell^{(i)}B = 0$$

and after division by ϕ_x, we have, provided $\phi_x \neq 0$,

$$\ell^{(i)}\left\{\frac{1}{\phi_x}U_{t''} + \left(\lambda^{(i)} + \frac{\phi_t}{\phi_x}\right)U_\phi\right\} + \ell^{(i)}B\frac{1}{\phi_x} = 0$$

But as $\dfrac{\phi_t}{\phi_x} = -\lambda^{(1)}$ and $\dfrac{1}{\phi_x} = x_\phi$ the above equation becomes

$$\ell^{(i)}\left\{x_\phi U_{t''} + \left(\lambda^{(i)} - \lambda^{(1)}\right) U_\phi\right\} + \ell^{(i)} B x_\phi = 0 \tag{5.6}$$

$$i=1: \qquad \ell^{(1)} U_{t''} + \ell^{(1)} B = 0 \tag{5.7}$$

$$i=2: \qquad \ell^{(2)}\left\{x_\phi U_{t''} + \left(\lambda^{(2)} - \lambda^{(1)}\right) U_\phi\right\} + \ell^{(2)} B x_\phi = 0 \tag{5.8}$$

We have that the vector U is continuous across $C_0^{(1)}$ and that $A(U)$ is also continuous across $C_0^{(1)}$. Also the derivatives with respect to t'' are continuous across $C_0^{(1)}$. However, the derivatives with respect to ϕ are discontinuous. We shall define the jump conditions across $\phi = 0$ as follows,

$$[U]_{\phi=0^-} - [U]_{\phi=0^+} = 0,$$

$$[U_{t''}]_{\phi=0^-} - [U_{t''}]_{\phi=0^+} = 0$$

$$[U_\phi]_{\phi=0^-} - [U_\phi]_{\phi=0^+} = \Pi(t'') = \begin{bmatrix}\Pi_1\\ \Pi_2\end{bmatrix} \qquad \text{(say)}$$

and

$$[x_\phi]_{\phi=0^-} - [x_\phi]_{\phi=0^+} = X(t'') \qquad \text{(say)}$$

Note here that $X(t'')$ is a scalar and is the jump condition in the Jacobian x_ϕ.

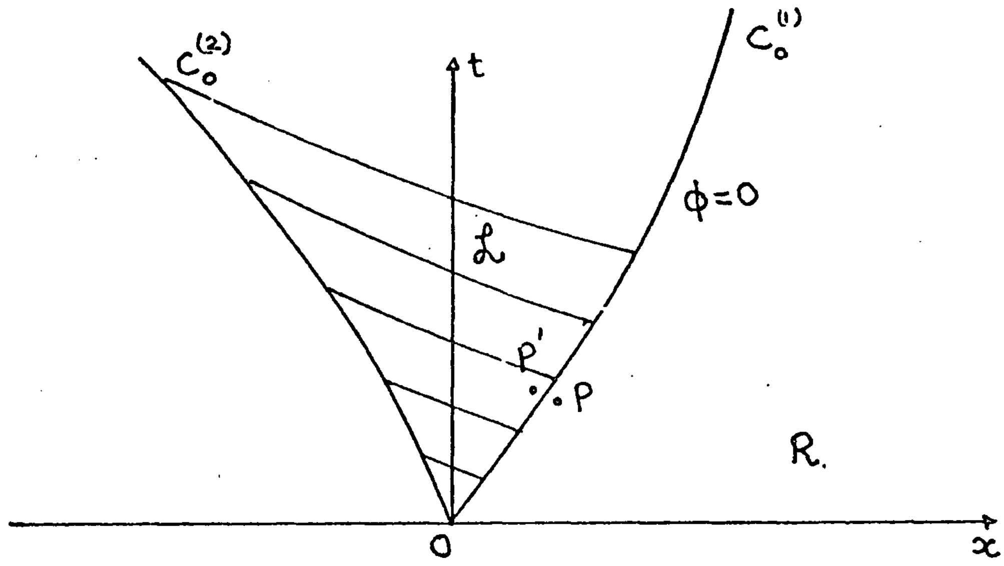


Figure 5.2

We also note that a constant solution $U = U_0$ exists in the region R satisfying the equation

$$U_{0t''} + A_0 U_{0x} + B_0 = 0 \qquad (5.9)$$

Since $l^{(1)} = 1,2$, (5.7) reduces to,

$$u_{t''} + 2c_{t''} + m = 0$$

and on differentiating with respect to ϕ , at a point in the region R, we have,

$$u_{\phi t''} + 2c_{\phi t''} = 0 \qquad (5.10)$$

There exists a similar equation for a point P' (say) in the region $\mathcal{L}$ behind $\phi = 0$ but in front of the backwards facing characteristic issuing out of the origin. Letting the points P and P' tend to a point on the line $\phi = 0$, differencing the equation across $\phi = 0$ and using the jump relations we obtain,

$$\Pi_{1t''} + 2\Pi_{2t''} = 0 \qquad (5.11)$$

where Π_1 and Π_2 are the components of Π .

Then differencing (5.8) across ϕ = 0 gives,

$$\ell^{(2)} U_{0t''} X + (\lambda_0^{(2)} - \lambda_0^{(1)}) \ell^{(2)} \Pi + \ell^{(2)} B_0 X = 0 \tag{5.12}$$

The suffix 0 denotes the values in the region of rest R . If we write $(x_\phi)_{\phi=0^+} = x_{o\phi}$, then on the side ϕ = 0^+ of the wave front the equation (5.8) becomes,

$$\ell^{(2)} x_{o\phi} U_{ot''} + (\lambda_0^{(2)} - \lambda_0^{(1)}) \ell^{(2)} U_{o\phi} + \ell^{(2)} B_0 x_{o\phi} = 0 \tag{5.13}$$

Multiplying (5.12) by $x_{o\phi}$ and subtracting the product of (5.13) and X then gives

$$-\ell^{(2)} U_{o\phi} X + \ell^{(2)} \Pi x_{o\phi} = 0 \tag{5.14}$$

Now, using the initial conditions u_0 = 0, (5.14) reduces to:

$$z(c_0)_x X + \Pi_1 - 2\Pi_2 = 0 \tag{5.15}$$

Since

$$x_{o\phi} = \frac{\partial x_0}{\partial \phi} , \quad c_{o\phi} = \frac{\partial c_0}{\partial \phi}$$

and

$$\frac{\partial c_0}{\partial \phi} \cdot \frac{\partial \phi}{\partial x_0} = (c_0)_x$$

we also have,

$$\frac{dx}{dt''} = \lambda^{(1)}$$

as the differential equation from which we may determine the wave front trace. Differentiating this with respect to ϕ , and working on $C_0^{(1)}$

$$\frac{\partial x}{\partial t''} = \lambda^{(1)},$$

and so

$$\frac{\partial}{\partial \phi}\left(\frac{\partial x}{\partial t''}\right) = \frac{\partial}{\partial \phi}\left(\lambda^{(1)}\right)$$

or

$$\frac{\partial}{\partial t''}\left(\frac{\partial x}{\partial \phi}\right) = \frac{\partial}{\partial \phi}\left(\lambda^{(1)}\right),$$

which may be written

$$\frac{\partial}{\partial t''}\left(x_\phi\right) = \frac{\partial}{\partial \phi}\left(\lambda^{(1)}\right) = \left(\nabla_u \lambda^{(1)}\right) U_\phi \qquad (5.16)$$

where ∇_u is the gradient operator with respect to the elements $u_1, u_2, \ldots\ldots u_n$ of U.

Differencing (5.16) across $C_0^{(1)}$, gives

$$\frac{dX}{dt''} = \left(\nabla_u \lambda^{(1)}\right)_0 \Pi \qquad (5.17)$$

and since $\lambda^{(1)} = u + c$

$$X_{t''} = \Pi_1 + \Pi_2 \qquad (5.18)$$

Integrating (5.17) along $C_0^{(1)}$ from 0 to $t' = \tau$ we get,

$$X(\tau) - X(0) = \int_0^\tau \left(\nabla_u \lambda^{(1)}\right)_0 \Pi \, dt'', \qquad (5.19)$$

but, as defined earlier,

$$X(\tau) = x_\phi(\tau)\Big|_{\phi=0^-} - x_\phi(\tau)\Big|_{\phi=0^+}$$

$$X(0) = x_\phi(0)\Big|_{\phi=0^-} - x_\phi(0)\Big|_{\phi=0^+}$$

The initial condition for ϕ is $\phi(x,0) = x$, and as $x_\phi = \frac{1}{\phi_x}$, we see that $X(0) = 1 - 1 = 0$.

As the state U_0 ahead of the disturbance is constant, so that the characteristics in U_0 are all parallel, we have that, $x_\phi(\tau)\Big|_{\phi=\sigma^+} = 1$. Thus the equation (5.19) can be written

$$x_\phi(\tau)\Big|_{\phi=\sigma^-} = 1 + \int_0^\tau \left(\nabla_u \lambda^{(1)}\right)_0 \Pi \, dt'' \tag{5.20}$$

The left hand side of (5.20) is just the Jacobian of the transformation evaluated immediately behind the wave front trace at time $t = \tau$. If, for some time $\tau = t_c$, the Jacobian vanishes, then the characteristics intersect and a discontinuity forms in the solution at the wave front. Hence, setting the Jacobian x_ϕ equal to zero in (5.20), and replacing τ by t_c, an equation is obtained from which, when it has a real non-negative solution, we can find t_c, the time of formation of a discontinuous solution,

$$0 = 1 + \int_0^{t_c} \left(\nabla_u \lambda^{(1)}\right)_0 \Pi \, dt'' \tag{5.21}$$

6. The breaking of waves

We can write the equations (5.11), (5.15) and (5.18), dropping the prime over the t without loss of generality, as

$$\Pi_{1t} + 2\Pi_{2t} = 0 \tag{6.1}$$

$$2(C_o)_x X + \Pi_1 - 2\Pi_2 = 0 \tag{6.2}$$

and

$$X_t = \Pi_1 + \Pi_2 \tag{6.3}$$

From (3.19) and (3.20),

$$(C_o)_x = -\frac{m}{2}\left(1 - \frac{m}{2}t\right)^{-1} \tag{6.4}$$

Writing $S = (C_o)_x$ we shall differentiate (6.2) with respect to t, giving

$$2S_t X + 2SX_t + \Pi_{1t} - 2\Pi_{2t} = 0$$

which reduces to,

$$S_t X + S(\Pi_1 + \Pi_2) + \Pi_{1t} = 0 \tag{6.5}$$

using (6.1) and (6.3).

Then we eliminate X between the equations (6.2) and (6.5) to obtain the equation,

$$2S\Pi_{1t} + (2S^2 - S_t)\Pi_1 + 2(S_t + S^2)\Pi_2 = 0 \tag{6.6}$$

If we differentiate (6.4) with respect to time, we have

$$S_t = \frac{m}{2}\left(1 - \frac{m}{2}t\right)^{-2}\left(-\frac{m}{2}\right)$$
$$= -S^2$$

Thus the equation (6.6) reduces to

$$\Pi_{1t} + \frac{3}{2} S \Pi_1 = 0 \tag{6.7}$$

Now integrate (6.7) with the initial condition $\Pi_1 = \tilde{\Pi}_1$ at $t = 0$, to find

$$\int_{\tilde{\Pi}}^{\Pi_1} \frac{\Pi_{1t}}{\Pi_1} dt = -\frac{3}{2} S = \int_0^t \frac{3}{2} \cdot \frac{m}{2} \left(1 - \frac{m}{2} t\right)^{-1} dt$$

Hence

$$\log\left(\frac{\Pi_1}{\tilde{\Pi}_1}\right) = -\frac{3}{2} \log\left(1 - \frac{m}{2} t\right) = \log\left[\left(1 - \frac{m}{2} t\right)^{-3/2}\right]$$

and we now write

$$\Pi_1 = \tilde{\Pi}_1 \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}}, \tag{6.8}$$

where $\tilde{\Pi}_1$ is the initial value of Π_1.

Then, by integrating (6.1), we get

$$\Pi_1 + 2\Pi_2 = \tilde{\Pi}_1 + 2\tilde{\Pi}_2,$$

where $\tilde{\Pi}_2$, is the initial value of Π_2.

whence

$$\Pi_2 = \tilde{\Pi}_2 + \frac{1}{2} \tilde{\Pi}_1 \left[1 - \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}}\right] \tag{6.9}$$

Since $\lambda^{(1)} = u + c$, (5.21) reduces to,

$$0 = 1 + \int_0^{t_c} \left(\Pi_1 + \Pi_2\right) dt \tag{6.10}$$

Now substituting for Π_1 and Π_2 from (6.8) and (6.9) in (6.10) we get the following equation

$$0 = 1 + \int_0^{t_c} \left\{ \tilde{\Pi}_1 \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}} + \tilde{\Pi}_2 + \frac{1}{2} \tilde{\Pi}_1 \left[1 - \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}}\right] \right\} dt \tag{6.11}$$

Also, by definition,

$$\Pi_1 = u_\phi \Big|_{\phi=0^-} - u_\phi \Big|_{\phi=0^+}$$

$$= u_\phi \Big|_{\phi=0^-} \quad \text{Since } u_\phi \Big|_{\phi=0^+} = 0,$$

and

$$\hat{\Pi}_1 = \underset{t\to 0}{\mathrm{lt}}\, \Pi_1 = \hat{u}_\phi = \hat{u}_x \hat{x}_\phi$$

so that

$$\hat{\Pi}_1 = \hat{u}_x \quad \text{as} \quad \hat{x}_\phi = 1$$

Similarly,

$$\Pi_2 = \underset{t\to 0}{\mathrm{lt}}\, \Pi_2 = \underset{t\to 0}{\mathrm{lt}} \quad C_\phi \Big|_{\phi=0^-} - C_\phi \Big|_{\phi=0^+}$$

$$= x_\phi \hat{C}_x + x_\phi \frac{m}{2}$$

$$= C_x + \frac{m}{2}$$

Thus (6.11) becomes

$$0 = 1 + \int_0^{t_c} \left[\frac{1}{2} \hat{u}_x \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}} + \frac{\hat{\Pi}_1}{2} + \hat{\Pi}_2 \right] dt$$

that is,

$$0 = 1 + \int_0^{t_c} \left[\frac{1}{2} \hat{u}_x \left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}} + \frac{1}{2} \hat{u}_x + \hat{C}_x + \frac{m}{2} \right] dt \qquad (6.12)$$

The equation (6.2) holds along $C_0^{(1)}$ and if we take the limit as $t \to 0$ along $\phi = 0^-$, we get

$$\hat{\Pi}_1 = 2\hat{\Pi}_2$$

that is

$$\frac{1}{2} \hat{u}_x = \hat{C}_x + \frac{m}{2}$$

so that we get,

$$0 = 1 + \int_0^{t_c} \left[2\hat{C}_x + m + \left(\hat{C}_x + \frac{m}{2}\right)\left(1 - \frac{m}{2} t\right)^{-\frac{3}{2}} \right] dt \qquad (6.13)$$

The solution of (6.13) depends on the value of $\hat{C}_x$, where $\hat{C}_x$ is results obtained by Stoker and Jeffrey are the same for a flat sea bed. But Jeffrey's method has the advantage that it may be applied to any problem having a sloping beach. For a disturbance given by (5.1), the critical time t_c and the corresponding distance x_c obtained by Stoker (25) are,

$$x_c = \frac{2c_o(c_o+u_o)^2}{3gA\omega} \tag{6.14}$$

and

$$t_c = \frac{2c_o(c_o+u_o)}{3gA\omega} \tag{6.15}$$

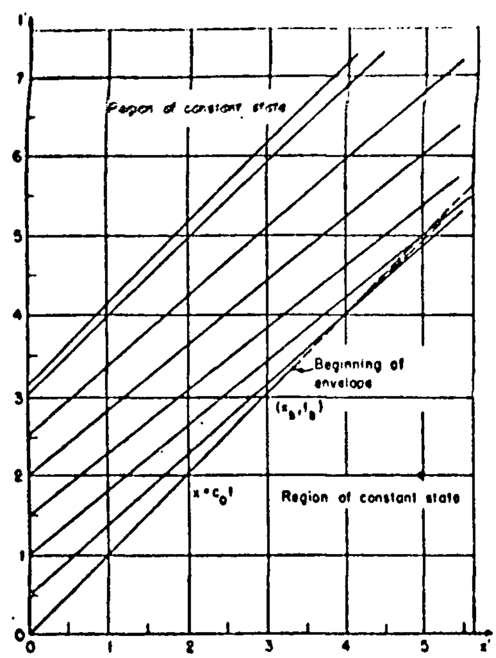
We also see from the definition of the surface wave speed $c = \sqrt{gh}$ that the propagation speed of a wave of height h increases with the height of a wave above the undisturbed level, and so it follows immediately that the higher points in the wave surface will propagate at a higher speed than the lower points in front of them, with the result that the crest of a wave overtakes the trough. Consequently, the wave becomes steeper and eventually the wave curls over and breaks. From equation (6.14) and (6.15) we see that the breaking depends on the amplitude and frequency. Hence the shorter the wave, the sooner it will break. Also the waves will break early if u_o is small, whilst if u_o is negative the breaking will be even sooner.

Another important deduction from the theory is that the maximum surface elevation of the waves propagating into still water is independent of time and distance. The position of the breaking point also depends on the type of wave that is propagated into still water.

We shall now discuss a few cases of the breaking of waves in shallow water of constant depth. The following table indicates the calculations made by Stoker (23) for three cases.

Case	Type of Pulse	A	Range of $t' = \omega\sqrt{gh}\,t$
1		.2	$0 \le t' \le \pi$
2		$-.2$	$0 \le t' \le 2\pi$
3		$-.1$	$0 \le t' \le 6\pi$

The case 1 is a half sine pulse in the form of a positive elevation. Fig. 6.1 shows the straight characteristics in the (x,t) plane. Here we observe that the envelope begins on the initial characteristic with two distinct branches which meet in a cusp at the breaking point (x_c, t_c) as given by (6.14) and (6.15).



The Figure 6.2 gives the shape of the wave for two different times. We see that the front of the wave steepens until it finally becomes vertical for $x = x_c$ and $t = t_c$, while the back of the wave flattens out.

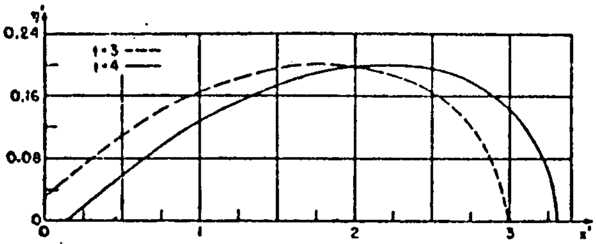


Figure 6.2. Wave height versus distance for a half sine wave of amplitude h_o in water of constant depth at 2 instants, where h_o = height of the still water level (23)

We also observe that the region between the two branches (of Figure 6.1) of the envelope is narrow and hence the influence of the developing breaker over the motion of the water behind it is very little. Hence we might be justified in assuming that the solution by characteristics as given by the Figure 6.2 is valid approximately for t just above t_c. Figure 6.3 refers to a wave at a time greater than t_c.

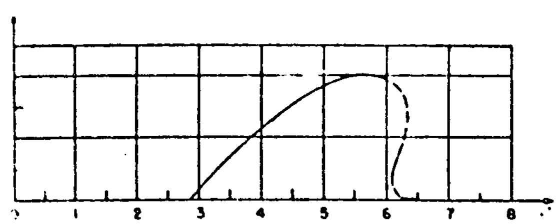


Figure 6.3 η' versus x' at t' = 6 for a non-sloping bottom, where the pulse is a half-sine wave. The dotted part of the curve represents η' in the region between the branches of the envelope (23).

Case 2 refers to a depression phase which preceeds a positive elevation.

Figure 6.4 which refers to case 2 shows that the envelope of the characteristics begins in the interior of the simple wave region and not on the initial characteristic.

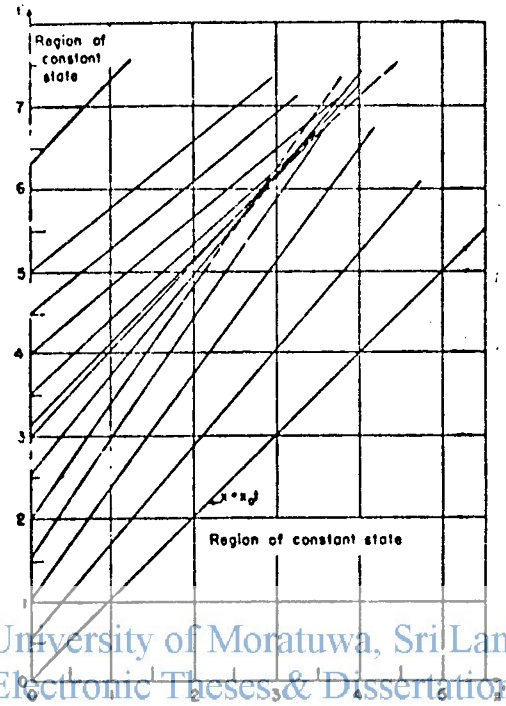


Figure 6.4 Characteristic diagram in the (x',t') plane (23)

Figures 6.5a and 6.5b show three stages of a wave propagating into still water.

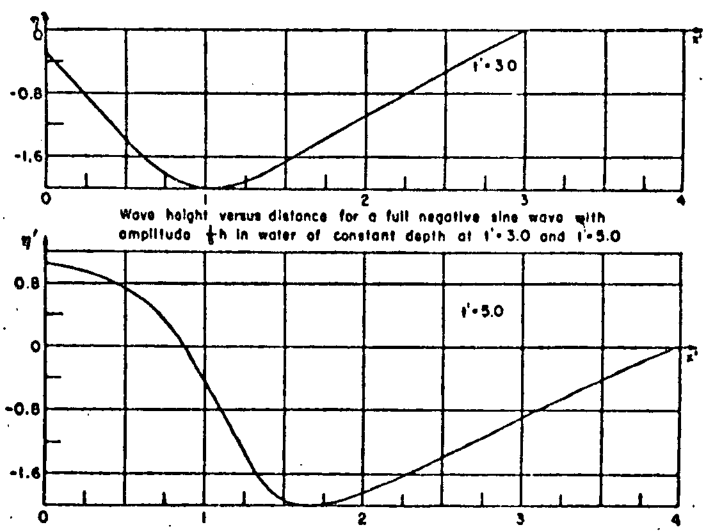


Figure 6.5a (23)

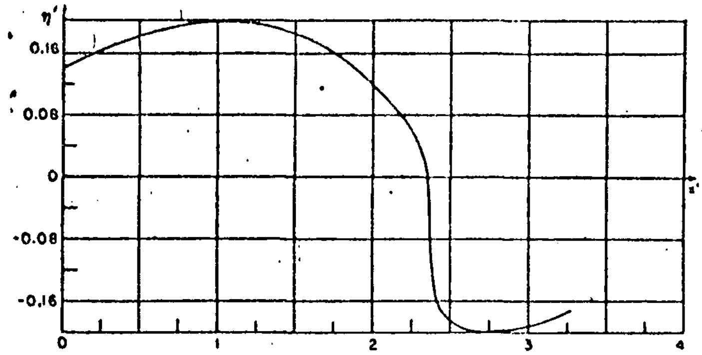


Figure 6.5b Wave height versus distance for a full negative sine wave with amplitude $\frac{1}{5}$ h in water of constant depth at t' = 6.28 (23)

It is seen that the steepening is very marked as the breaking point is reached. Figure 6.7 shows the shape of the wave just after passing the breaking point. Near the breaking point the curvature of the water surface is large. However, shallow water theory is accurate only for smaller curvatures. Thus we should revert to the original exact formulation of the problem in terms of a potential function with a non linear free surface if a solution to this question is required.

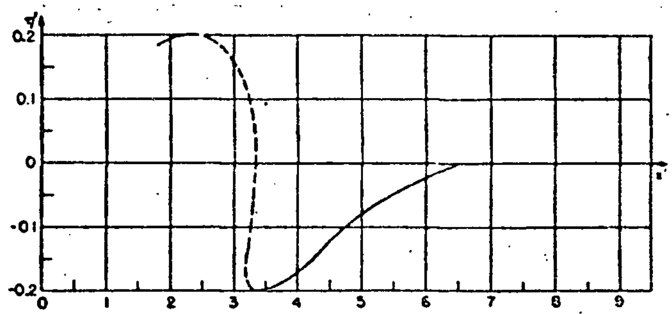


Figure 6.7 η' versus x' at t' = 7 for non-sloping bottom where the pulse is an entire negative sine-wave. The dotted part of the curve represents η' in the region between the branches of the envelope (23)

Now we shall consider some cases of breaking on a uniformly sloping beach. Here too the principle is the same as that for waves in water of constant depth, but with some differences, such as the fact that the amplitude of a progressing wave increases and its wave length decreases as it moves towards the shore. This implies that early breaking is possible by having a steep beach. In these cases the characteristics are not straight lines in the region of the (x,t) plane bordering the region of constant state. Also the velocity and the displacement of the water surface are not constant along the characteristics. Thus we are forced to integrate the differential equations numerically. From (3.20) we have the initial characteristic as,

$$x = t - \frac{1}{4} m t^2,$$

which is a parabola with its vertex at a point with the x co-ordinate corresponding to the shore line.

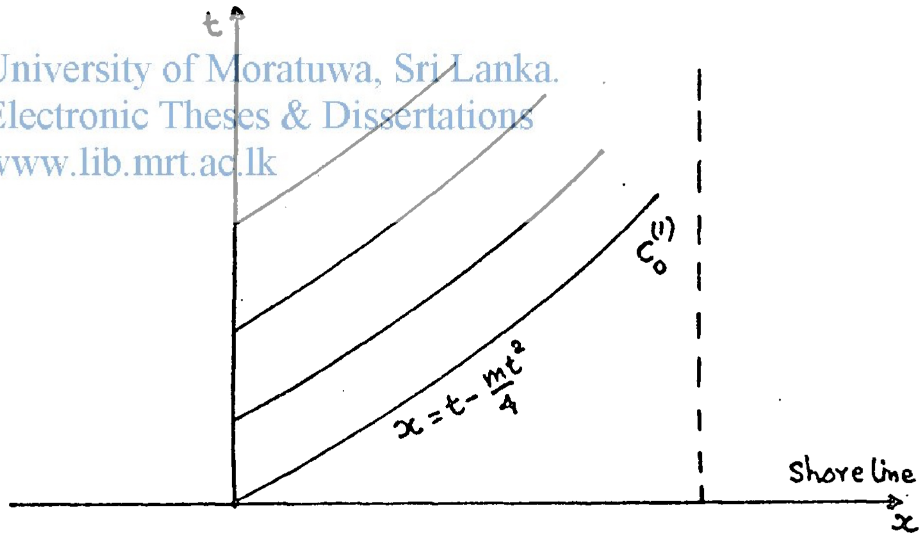


Figure 6.8 Characteristics for a uniformly sloping beach in the (x,t) plane (23)

In order to obtain the remaining characteristics, we make use of the method of finite differences. A set of points on the initial characteristic $C_0^{(1)}$ is taken at equal time intervals, the size of the interval is chosen depending on the accuracy required. Along $C_0^{(1)}$, $u = 0$, x,t are known and hence C is known. Along $C^{(2)}$ between points (2,2) and (1,2) we have

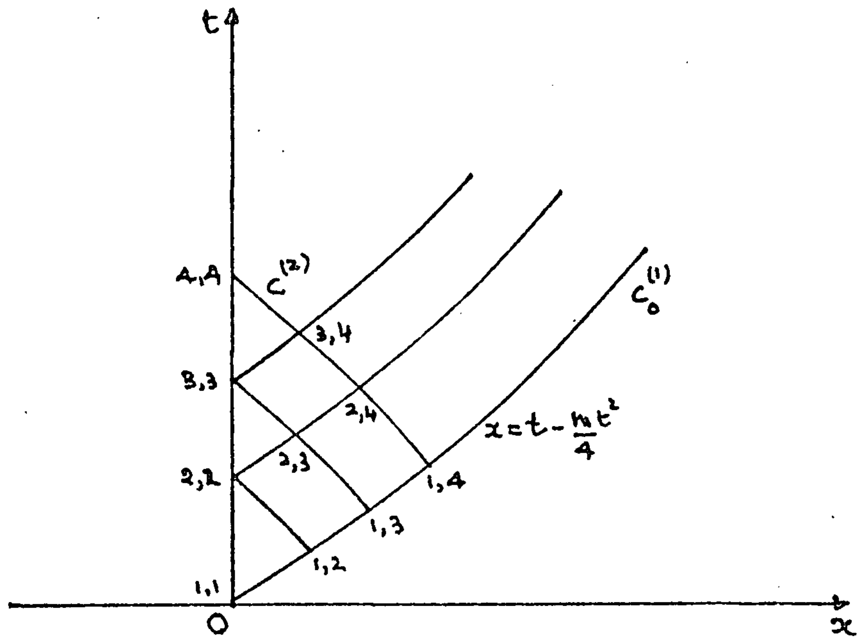


Figure 6.9 (23)

from (3.24)

$$u - 2c + mt = mt_{1,2} - 2c_{1,2} \tag{6.16}$$

If the arc $C^{(2)}$ between point (1,2) and (2,2) is sufficiently short then we can replace the differential equation $\frac{dx}{dt} = u - c$, which is valid along the above arc approximately by,

$$\frac{-x_{1,2}}{t_{2,2} - t_{1,2}} = \frac{1}{2}\left\{-c_{1,2} + (u_{2,2} - c_{2,2})\right\}. \tag{6.17}$$

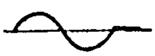
Since $C^{(2)}$ passes through point (2,2), from (6.16) we have

$$u_{2,2} = mt_{1,2} - 2c_{1,2} + 2c_{2,2} - mt_{2,2} \tag{6.18}$$

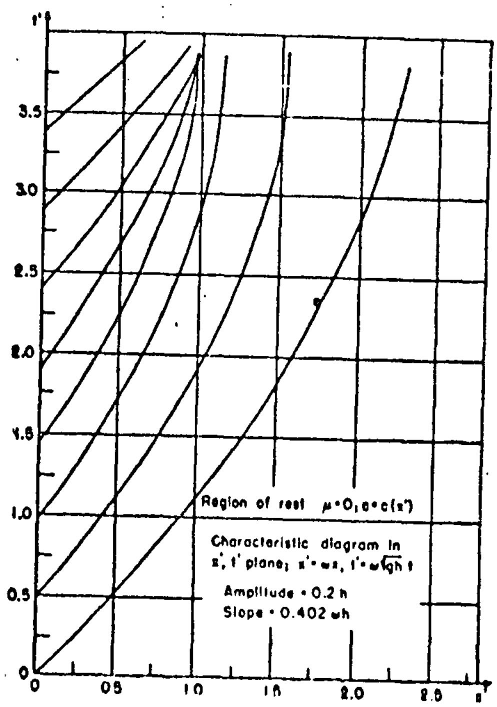
From the initial conditions $c_{2,2}$ may then be obtained. Thus we can find all the relevant quantities at the point (2,2). This procedure is adopted to determine the positions (2,2), (3,3), (4,4) etc. on the t axis.

This method is applicable to all net points in the interior region between C_o''' and the t axis.

The table below gives some of the numerical calculations made by Stoker for the cases indicated.

Case	Type of Pulse	Amplitude	Slope	Breaking Point $t' = \omega\sqrt{gh}\,t$	Breaking Point $x' = \omega x$	Increase in Amplitude at Breaking
1		$.2h$	$.4\omega h$	4	0.8	$\simeq 60\%$
2		$.08h$	$.2\omega h$	7	3.0	$\simeq 50\%$
3		$.02h$	$.4\omega h$	20	14.0	$\simeq 30\%$

We see from the above table that (as stated earlier) the breaking occurs earlier with increase in amplitude and decrease in wave length. In Fig. 6.10, the set of characteristics shown are calculated by the method of finite differences for case 1.



The time interval is $\Delta t' = 0.5$ and the Fig. 6.11 shows the shape of the wave surface for $t' = 3$ and $t' = 4$.

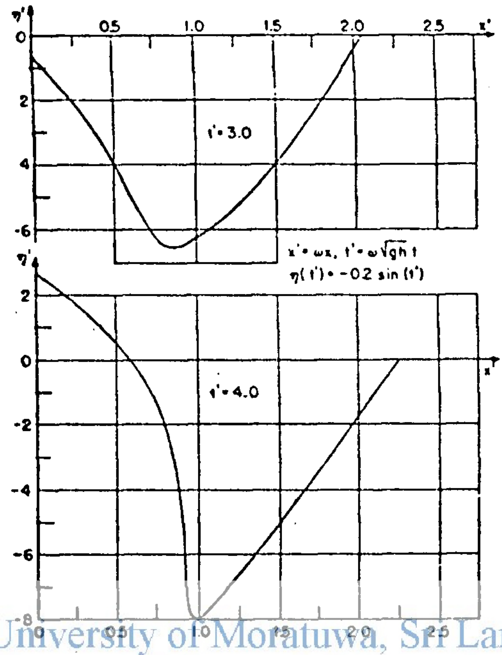
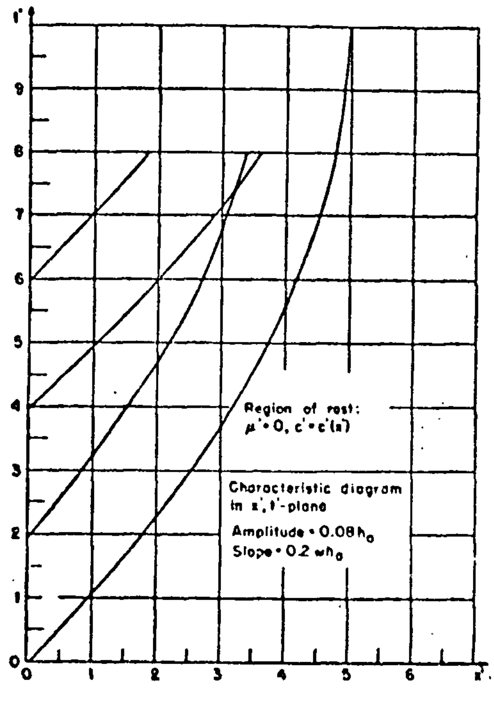


Figure 6.11 Wave height η' versus x' at two different instants. (23)

Then the Figures 6.12 and 6.13 show the characteristics in the (x',t') plane and the wave surface for two different times for case 2 in the table.



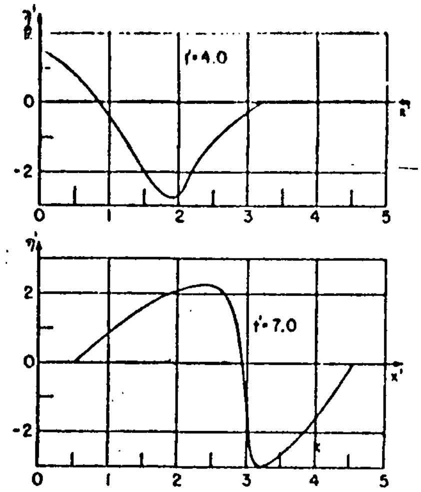


Figure 6.13 (23)

The wave surface in case 3 at two different times is shown in Figure 6.14. We see that the wave length shortens, the amplitude increases and the wave front steepens as the train of waves moves towards the shore. But these changes are very small. In case 3 where the amplitude is sufficiently small the shape of the wave could be obtained fairly accurately by using the linear shallow water theory.

Figure 6.15 shows the results obtained by using the two theories.

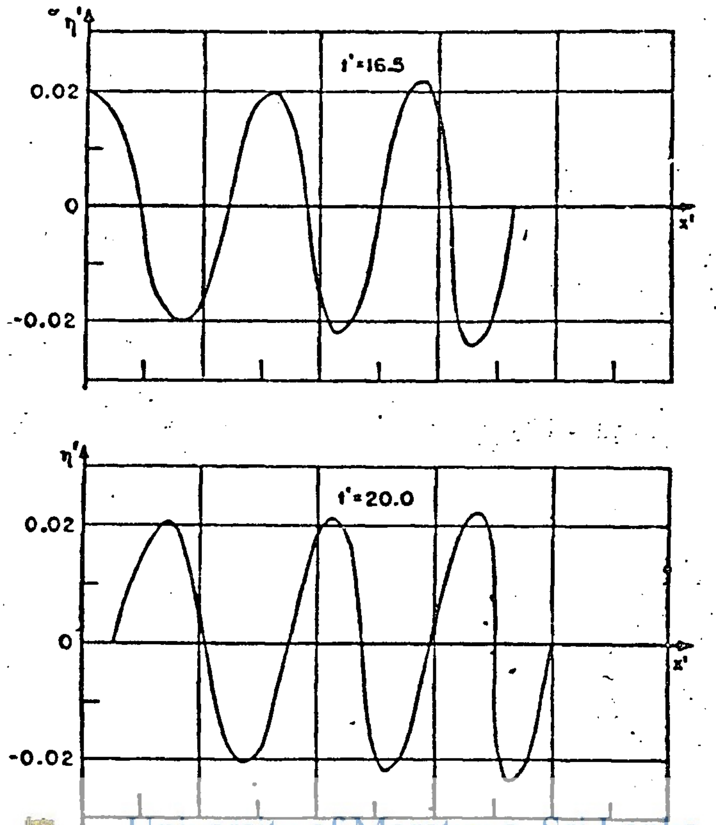


Figure 6.14 (22)

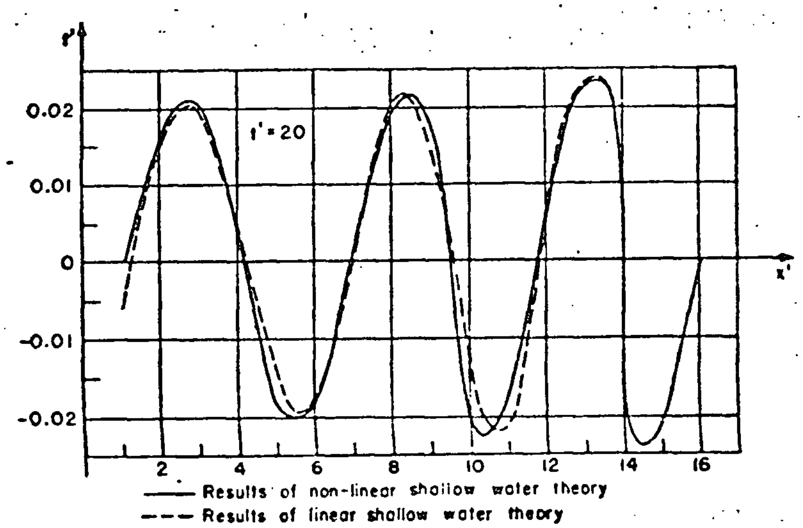


Figure 6.15 Comparison of linear and non linear shallow water theorics (22)

The Bore

We shall now consider the water profile after breaking. Stoker (24) and Biesel (5) suggested some methods of solving the differential equations after breaking. Stoker suggested the form of curves given in Figure 6.16.

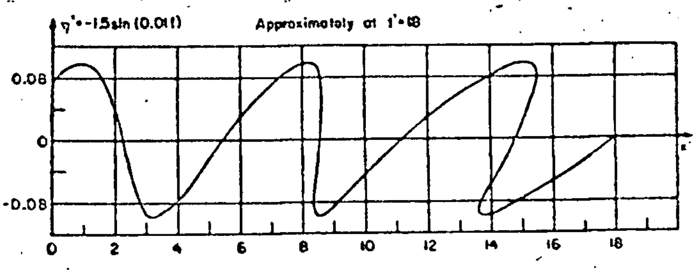


Figure 6.16 (23)

The results of Biesel's work using a perturbation procedure can also be shown graphically by Figure 6.17a, 6.17b, 6.17c and 6.17d.

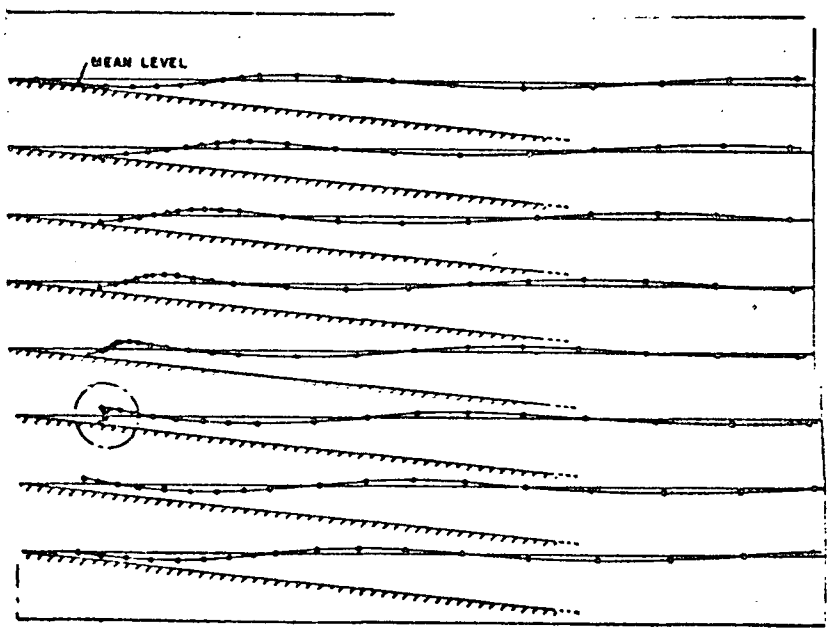


Figure 6.17a Progression and breaking of a wave on a beach of 1 in 10 slope. First order theory. (23).

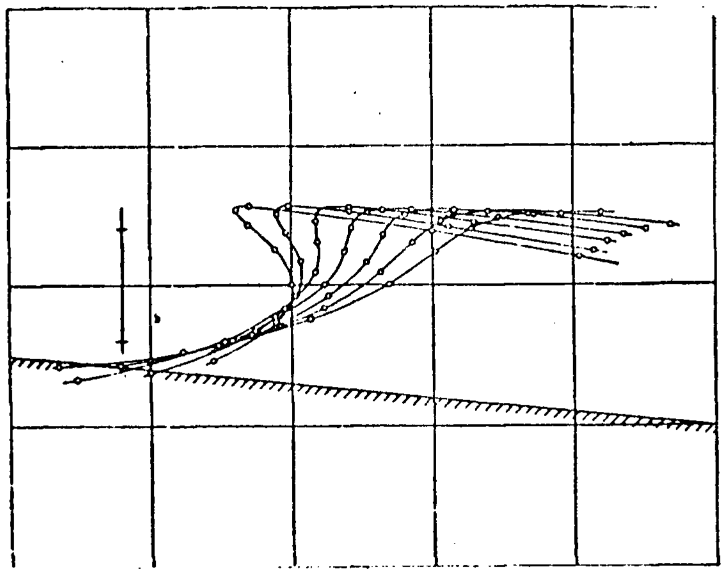
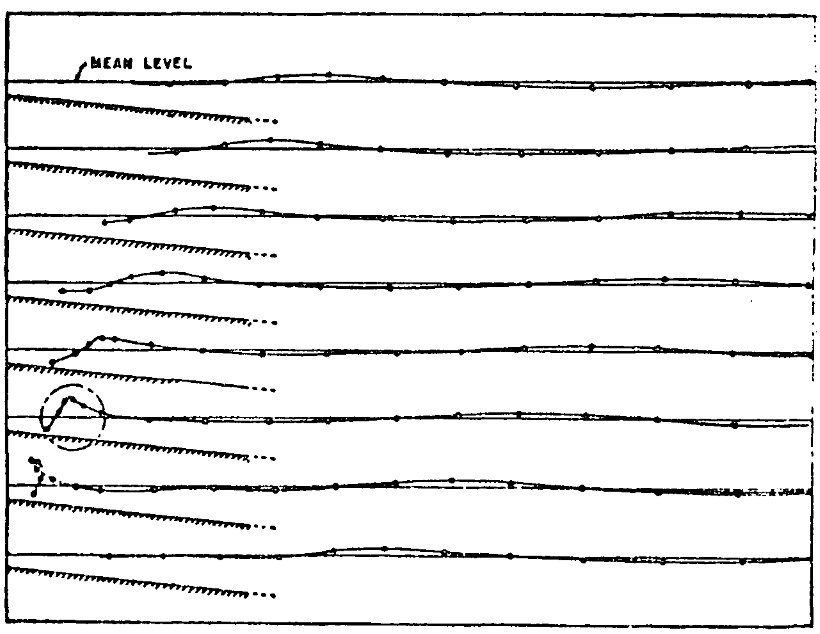


Figure 6.17b Details of breaking of wave shown in Fig. 6.17a. First order theory. (23)

The Figures 6.17a and 6.17b show the results when the theory is carried out to the first order only, and the Figures 6.17c and 6.17d refer to results when the theory is carried out to second order terms. It is seen that if second order terms are taken the breaking seems to occur earlier.



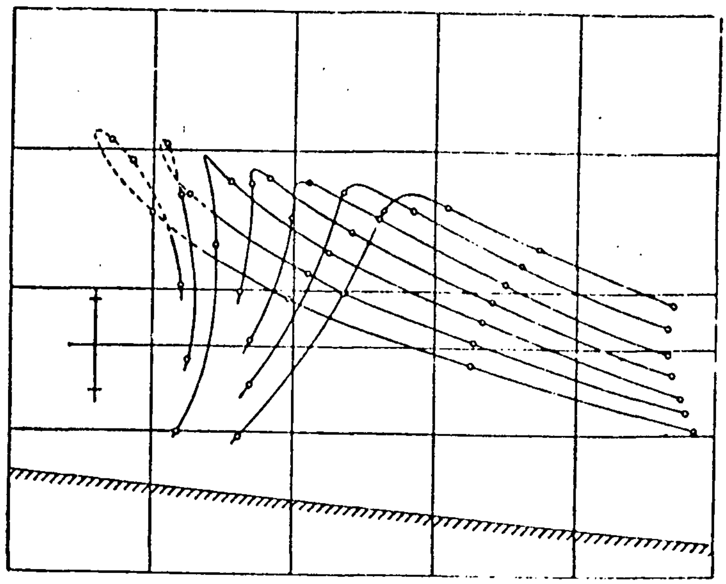


Figure 6.17d Details of breaking of wave shown in Fig. 6.17c. Second-order theory. (22)

and the height of the wave at breaking is greater. This shows that we cannot expect to get a good approximation to the wave profile near the breaking point using the shallow water theory.

In open channels sometimes a situation arises when a steady progressing wave front which is steep and turbulent is created as shown in Figure 6.18.

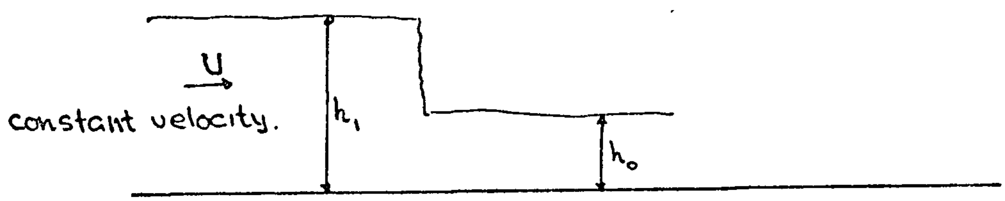


Figure 6.18 (22)

If the discontinuous front is a moving one it is called a bore and if it is stationary a hydraulic jump.

Simple explanation of bore formation in a sloping stream

Consider water of depth h flowing with velocity u in a channel whose bed is inclined downwards at a small angle α to the horizontal.

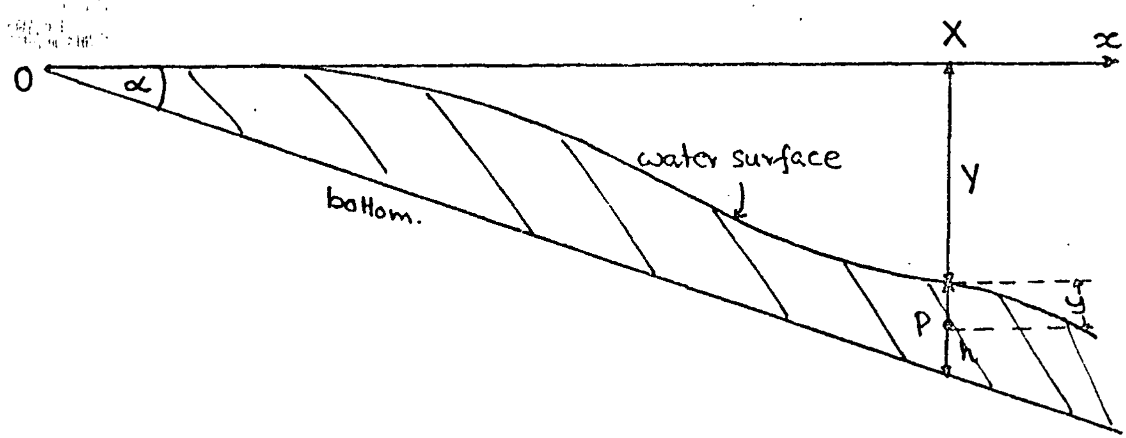


Figure 6.19

Let OX = x, and let y be the depth of a point P in the water with surface at depth Y below the origin O as in Figure 6.19.

Bernoulli's equation then gives,

$$\frac{p}{\rho} + \frac{1}{2}u^2 - g(y+Y) = \text{Constant.} \qquad (6.19)$$

where $p = \rho gy,$ (6.20)

ρ = water density

u = water speed

g = acceleration due to gravity

giving

$$\frac{1}{2}u^2 - gY = \text{Constant.} \qquad (6.21)$$

If α is small we can write

$$Y + h = \alpha x, \qquad (6.22)$$

So that (6.21) becomes

$$\frac{1}{2}u^2 + gh = \text{Constant} + x\alpha g \tag{6.23}$$

Now continuity of flow implies that

$$uh = A = \text{constant} = \text{initial value of product} \tag{6.24}$$

Hence (6.23) can be written

$$\frac{1}{2}\frac{A^2}{h^2} + gh = B + x\alpha g,$$

with B a constant.

Hence

$$\frac{dh}{dx} = \alpha g \Big/ \left(g - \frac{A^2}{h^3}\right) = \alpha \Big/ \left(1 - \frac{u^2}{gh}\right).$$

Thus, if $u^2 < gh$, the depth h of the water increases with distance, and if $u^2 > gh$, the depth h of the water decreases with distance. If $u^2 \to gh$ the above argument no longer holds and the slope of the surface would become large giving rise to a possible mechanism for bore formation.

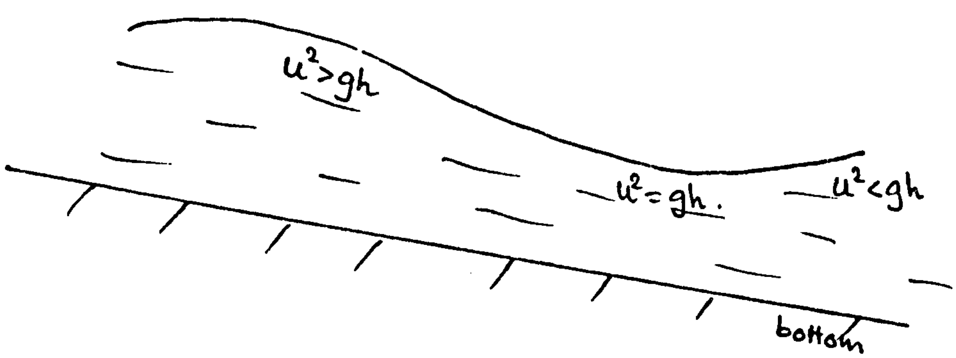


Figure 6.20

<u>Relations across a strong discontinuity line</u>

We have seen earlier in our discussion that in general, a quasi-linear hyperbolic equation will not have a differentiable solution which is unique for all time. This is also true for a system of such equations.

The non-uniqueness of a solution at a given point suggests the introduction of discontinuous solutions to overcome this differentiability problem. That is we try to piece together a solution from differentiable solutions which are discontinuous across a line and to determine the nature of such a discontinuity. This corresponds to a shock in gas dynamics or to a bore in water waves.

Now consider the following equation (6.25) written in the divergence or conservation form,

$$\frac{\partial F}{\partial t} + \operatorname{div} G = H \tag{6.25}$$

where F, G, H are functions at x,t and $U = [u_1, u_2 \cdots u_n]^T$. Let us assume that U is discontinuous across line L in the (x,t) plane (Figure 6.21) and that the equation (6.25 is valid in some region R with

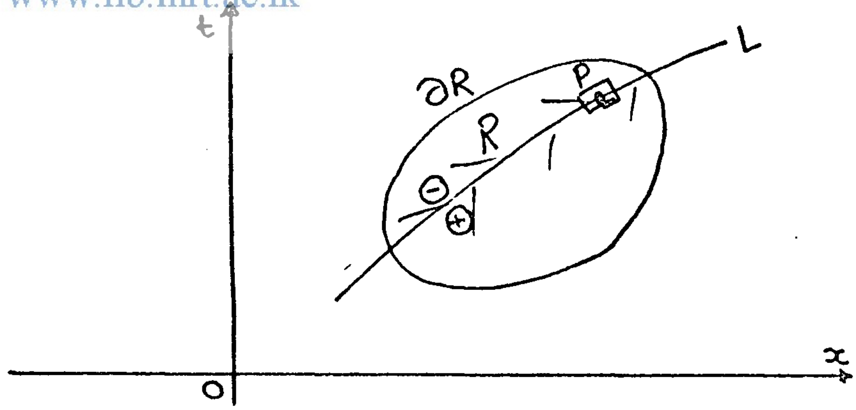


Figure 6.21

boundary ∂R that is traversed by L. We have

$$\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x} = H, \tag{6.26}$$

So let us now integrate (6.26) over R to obtain

$$\iint_R \left(\frac{\partial F}{\partial t} + \frac{\partial G}{\partial x}\right) dx\,dt = \iint H \, dx\,dt \tag{6.27}$$

Now apply Green's Theorem to the left hand side of (6.27) and we get,

$$\oint_{\partial R} (F\,dx - G\,dt) = \iint_R H\,dx\,dt. \qquad (6.28)$$

Allowing R to shrink to zero about P, we see that the right hand side of (6.28) tends to zero. (We assume that H is finite). (6.28) becomes after division by dt,

$$\left[F\frac{dx}{dt} - G\right]_{\text{jump across L}} = 0 \qquad (6.29)$$

Setting $\frac{dx}{dt} = \hat{\lambda}$ then gives the result

$$(F^-\hat{\lambda} - G^-) - (F^+\hat{\lambda} - G^+) = 0$$

or

$$[[F]]\hat{\lambda} = [[G]] \qquad (6.30)$$

where $[[\,.\,]]$ denotes the jump across the line L and $\hat{\lambda}$ is the slope of L at P. (6.30) is a system of algebraic equations. In gas dynamics a system of equations of this type is called the Rankine-Hugoniot equations. Let us now write the equation (33) and (34) in conservation form. We see by inspection that (33) can be written as

$$\frac{\partial}{\partial t}(u) + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2 + c^2 - Y\right) = 0 \qquad (6.31)$$

Equation (3.4) is not in divergence form as it stands. It can be brought into this form by multiplying by C, when it may be written

$$\frac{\partial}{\partial t}(c^2) + \frac{\partial}{\partial x}(uc^2) = 0 \qquad (6.32)$$

Combining (6.31) and (6.32) we have the system,

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ c^2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \frac{1}{2}u^2+c^2-Y \\ uc^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad (6.33)$$

and in terms of our previous notation,

$$F = \begin{bmatrix} u \\ c^2 \end{bmatrix}, \quad G = \begin{bmatrix} \frac{1}{2}u^2+c^2-Y \\ uc^2 \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus the jump condition

$$[\![F]\!]\hat{\lambda} - [\![G]\!] = 0$$

takes the form

$$\left.\begin{aligned} [\![u]\!]\hat{\lambda} &= [\![\tfrac{1}{2}u^2+c^2-Y]\!] \qquad &(a) \\ \text{and} \qquad [\![c^2]\!]\hat{\lambda} &= [\![uc^2]\!] \qquad &(b) \end{aligned}\right\} \qquad (6.34)$$

The conditions (6.34) are algebraic conditions and they determine the behaviour of u and c across the "bore" and relate them to its speed of propagation $\hat{\lambda}$.

Let us apply this result to a wave advancing in water at rest over a step change of depth in a channel as shown in Fig. 6.22. Equation (6.34a) yields

$$c_+^2 - Y_+ = c_-^2 - Y_-$$

which gives us no new information as each side of the equation is identically zero.

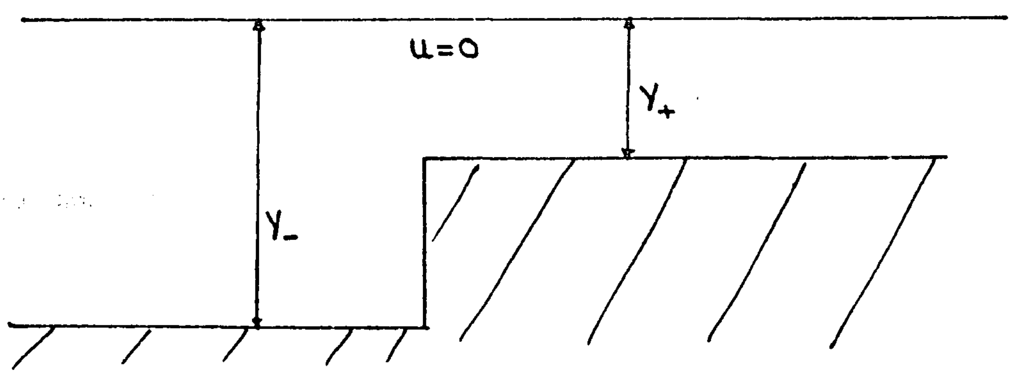


Figure 6.22

Equation (6.34b) becomes

$$(c_+^2 - c_-^2)\hat{\lambda} = 0$$

so that if $C_+ \neq C_-$, then $\hat{\lambda}$ must be zero. This means that the disturbance is stationary as would be expected from the nature of the problem.

7. Waves of finite amplitude on a sloping beach

The well established shallow water theory due to Stoker has been used by many authors to explain various problems. Carrier and Greenspan (7) used it on water waves of finite amplitude on a sloping beach. In this paper the authors assume a constant slope beach and the characteristic length ℓ_0 in the transformation is chosen depending on the problem.

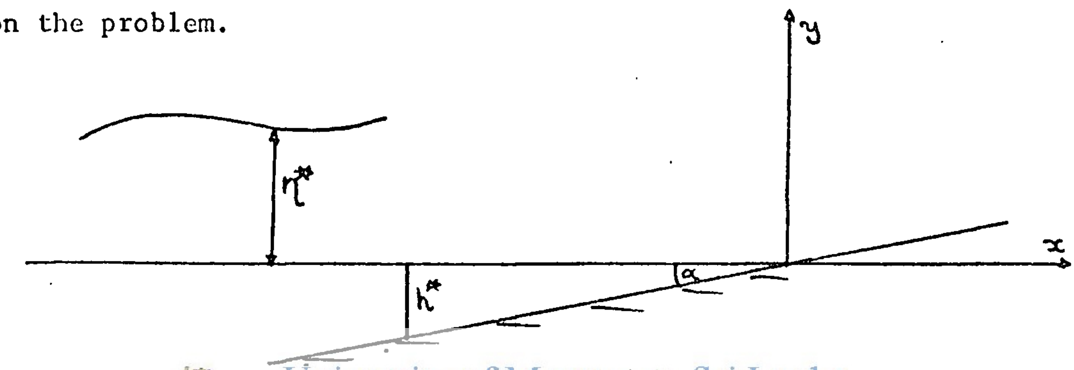


Figure 7.1 (7)

Having introduced a potential function $\phi(\sigma,\lambda)$ such that the horizontal velocity v is given by

$$v = \sigma^{-1}\phi_\sigma(\sigma,\lambda), \tag{7.1}$$

the authors obtained the following relationships,

$$x = \phi_\lambda/4 - \sigma^2/16 - v^2/2 \tag{7.2}$$

$$\eta = c^2 + x = \phi_\lambda/4 - v^2/2. \tag{7.3}$$

$$t = \lambda/2 - v \tag{7.4}$$

and

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0 \tag{7.5}$$

where η is the free surface elevation ...

from the origin and σ , λ are a pair of independent variables.

The free boundary line is the line $\sigma = 0$ in the (σ, λ) plane. Once the function $\phi(\sigma,\lambda)$ is chosen to suit the problem, then η , v, x, t are defined in terms of σ and λ .

If the Jacobian $\frac{\partial(x,t)}{\partial(\sigma,\lambda)}$ does not vanish in $\sigma > 0$, the solution $\eta(x,t), v(x,t)$ is single valued and hence the waves do not break. The form of the function ϕ satisfying the above equations is given by

$$\phi = A J_0(\omega\sigma)\cos(\omega\lambda - \psi) \tag{7.6}$$

J_o being the Bessel Function. For a function given by (7.6) the Jacobian $\frac{\partial(x,t)}{\partial(\sigma,\lambda)}$ does not vanish in $\sigma > 0$ for $A \leq 1$.

The authors considered two initial value problems. The first example is a one parameter family of wave forms at t = 0 given by

$$\eta = \varepsilon\left[1 - \frac{5}{2}\frac{a^3}{(a^2+\sigma^2)^{3/2}} + \frac{3}{2}\frac{a^5}{(a^2+\sigma^2)^{5/2}}\right], \tag{7.7}$$

$$x = -\frac{\sigma^2}{16} + \varepsilon\left[1 - \frac{5}{2}\frac{a^3}{(a^2+\sigma^2)^{3/2}} + \frac{3}{2}\frac{a^5}{(a^2+\sigma^2)^{5/2}}\right], \tag{7.8}$$

where $a = \frac{3}{2}(1 + 0.9\varepsilon)^{\frac{1}{2}}$ and ε is a constant to be chosen.

The authors obtained the following equations for the motion of the instantaneous shore line

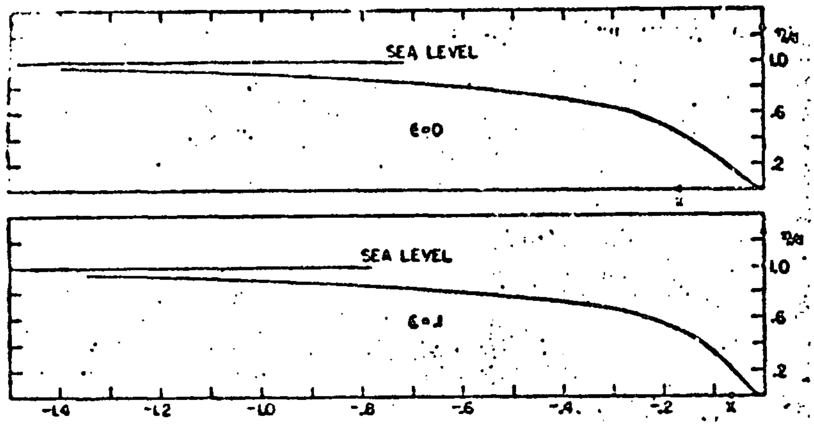


Figure 7.2 Initial wave shapes given by equations (7.7) and (7.8) for $\varepsilon \to 0$ and $\varepsilon = 0.1$ (7)

By setting $\sigma = 0$, we find that

$$v = \frac{8\varepsilon\lambda^2(5-\lambda^2)}{a(1+\lambda^2)^4} \tag{7.9}$$

$$x = -\frac{1}{2}v^2 + \varepsilon - \frac{\varepsilon}{(1+\lambda^2)^3}(1+3\lambda^2-2\lambda^4). \tag{7.10}$$

and

$$t = \frac{1}{2}a\lambda - v. \tag{7.11}$$

The maximum penetration distance is obtained by setting $v = 0$. It occurs when $\lambda^2 = 5$, when

$$x_{max} = 1.157\varepsilon. \tag{7.12}$$

The time history of the wave motion for $\varepsilon = 0.2$ is shown in Figures 7.3, 7.4, 7.5 and 7.6.

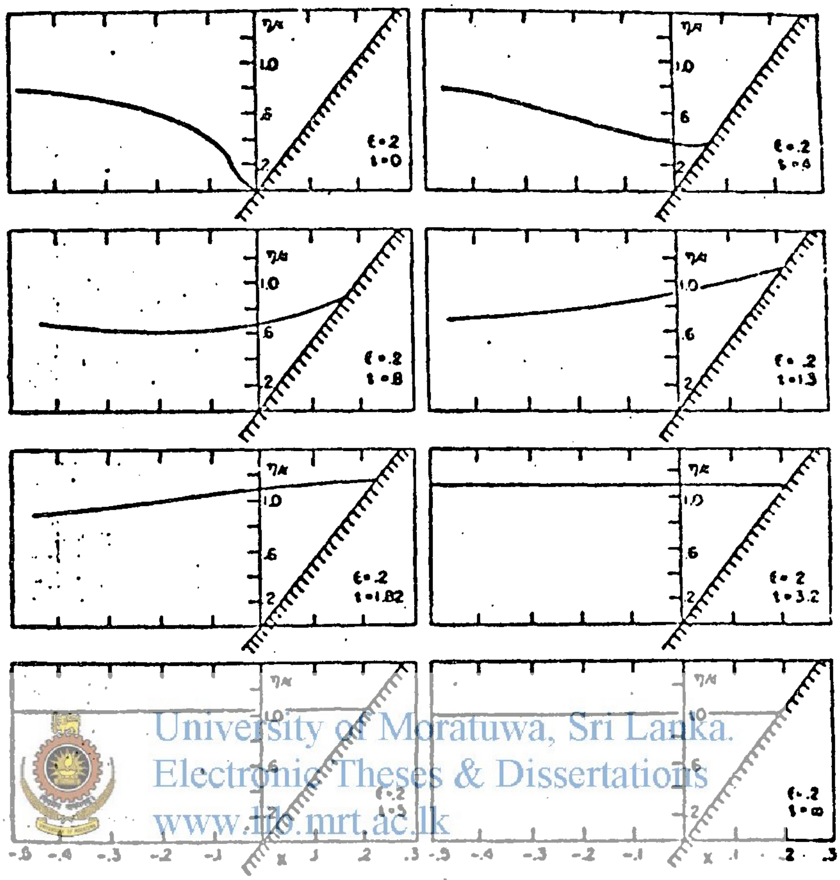


Figure 7.3 Time history of the wave form of equation (7.7) for ϵ = 0.2 near the coast line. (7)

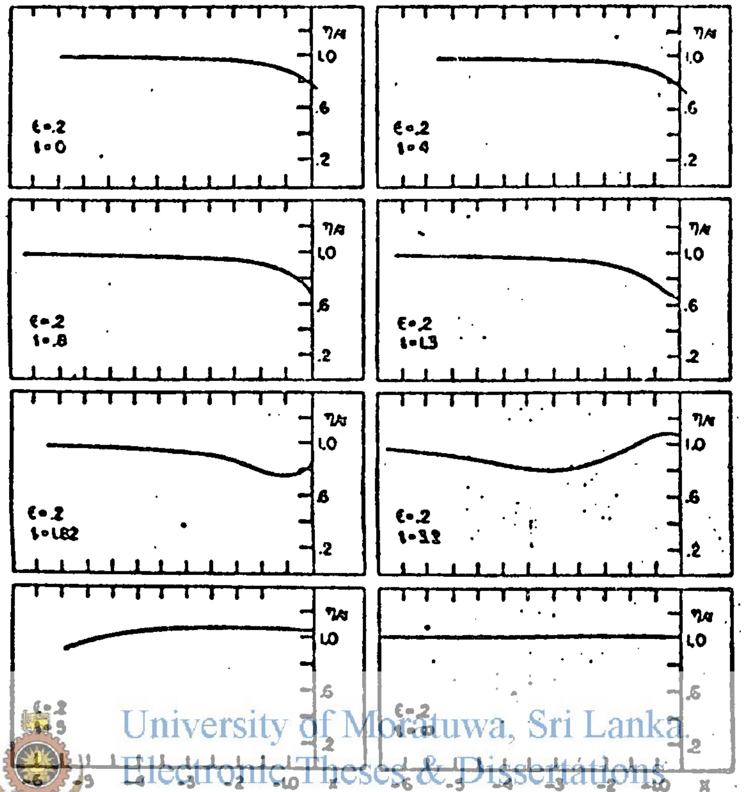


Figure 7.4 Time history of the wave-form of equation (7.7) for ϵ = 0.2, far from the coast line. (7)

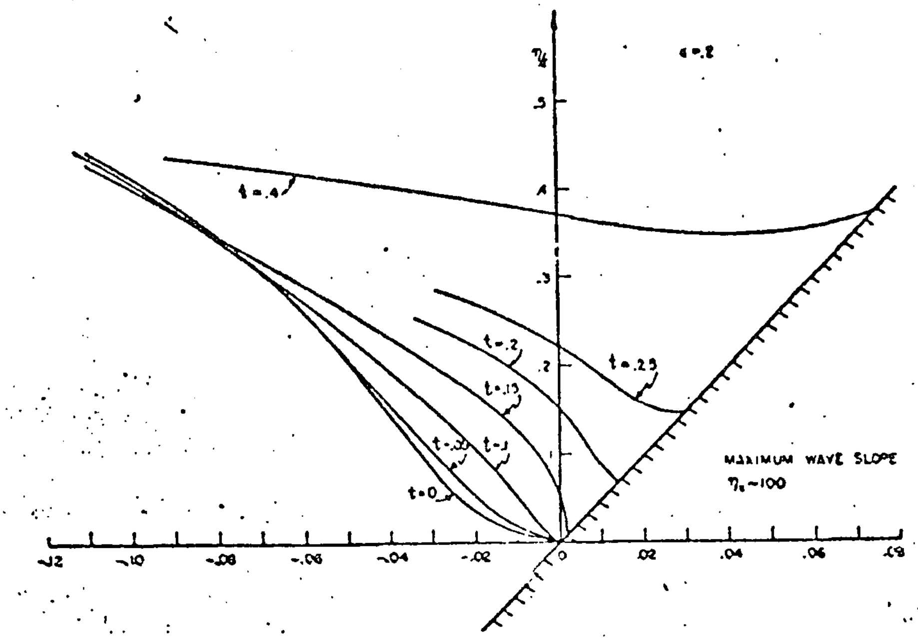


Figure 7.5 Time history of the wave-from of equation (7.7), for ϵ = 0.2, in the neighbourhood of the beach. (7)

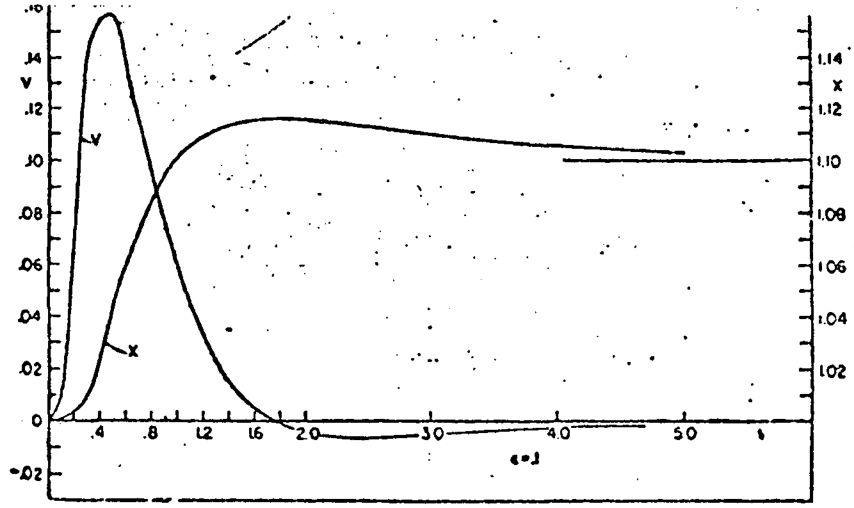


Figure 7.6. Coastline position and velocity versus time for the wave-form of equation (7.7) with ϵ = 0.1. (7)

We note that, as shown in Figure 7.3, the instantaneous shore line rises above the mean sea level and then slowly settles back. Further, there are no oscillations about the mean sea level and the waves do not break for $\varepsilon < 0.23$. In the second example (7) the motion of the stationary mound of water released at t = 0 is assumed to be given by,

$$\eta = \frac{1}{4}\varepsilon p^2 \sigma^4 e^{2-\sigma^2 p} \qquad (7.13)$$

and

$$x = \frac{1}{4}\varepsilon p^2 \sigma^4 e^{2-\sigma^2 p} - \frac{\sigma^2}{16} \qquad (7.14)$$

where

$$8p(1+\varepsilon) = 1.$$

This wave form is shown in Figure 7.7.

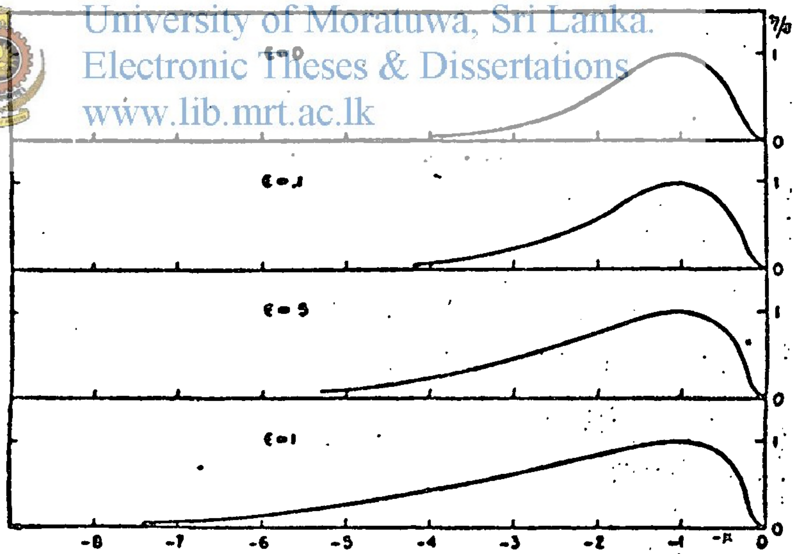


Figure 7.7 Exponential wave-forms of equation (7.14), ε = 0, 0.1, 0.5, 1. (7)

We see that the waves have zero slope at the origin and the initial maximum height is at a fixed position from the shore line. The authors were able to show that the quantities v, v_λ, v_σ are bounded and that the upper bounds are independent of σ and λ, so that,

$$|v| < M\varepsilon, \quad |v_\lambda| < M_1\varepsilon, \quad |v_\sigma| < M_2\varepsilon,$$

where M, M_1, M_2 are constants.

For sufficiently small ϵ , the Jacobian

$$|J| = \frac{\sigma}{4}\left|v_\sigma^2 - \left(\frac{1}{2} - v_\lambda\right)^2\right|$$
$$\doteq \frac{\sigma}{16} \neq 0$$

Hence the Jacobian does not vanish in the interior of the fluid showing that the waves given by the equations (7.13) and (7.14) do not break as they climb the shore. The Jacobian is zero only for σ = 0. But this is a property of the transformation and not in any way related to the initial wave shape. In this problem too (cf. Figures 7.8 and 7.9) the shore line motion is such that it first rises to a maximum height

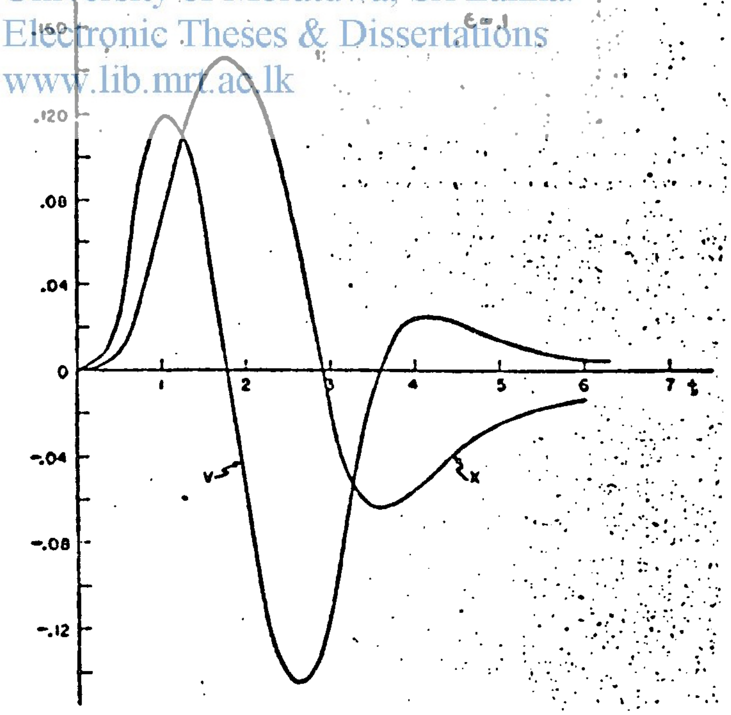


Figure 7.8 Coastline position and velocity versus time for the exponential wave ϵ = 0.1 of equation (7.14). (7)

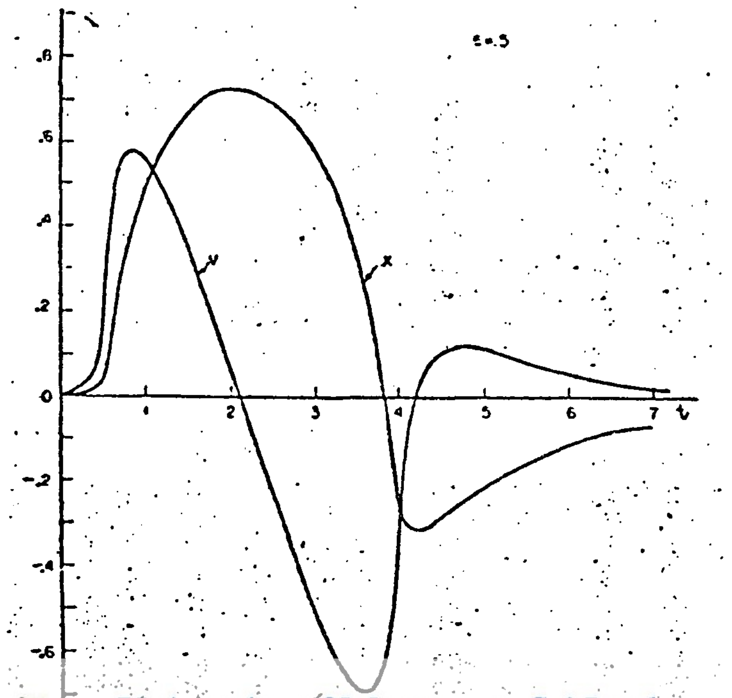


Figure 7.9 Coastline position and velocity versus time for the exponential wave ε = 0.5 of equation (7.14). (7)

and then falls back to a minimum below the rest position and then settles to the original mean sea-level slowly. Further, there are no oscillatory motions, and the maximum penetration distance obtained by setting v = 0 for λ = 2.41 is

$$x_{max} = 1.451\varepsilon \qquad (7.15)$$

Thus the maximum penetration distance in the second example is greater than in the first one for a particular ε. The authors have shown that there are progressive waves with positive amplitudes which do not break as they climb a sloping beach, even though the value of ε , and hence the initial shape of the wave, determines the breaking.

Greenspan (11) using the non-linear shallow water theory obtained the same basic equations as Stoker. The wave velocity was obtained as $C = (1 - x + \eta)^{\frac{1}{2}}$.

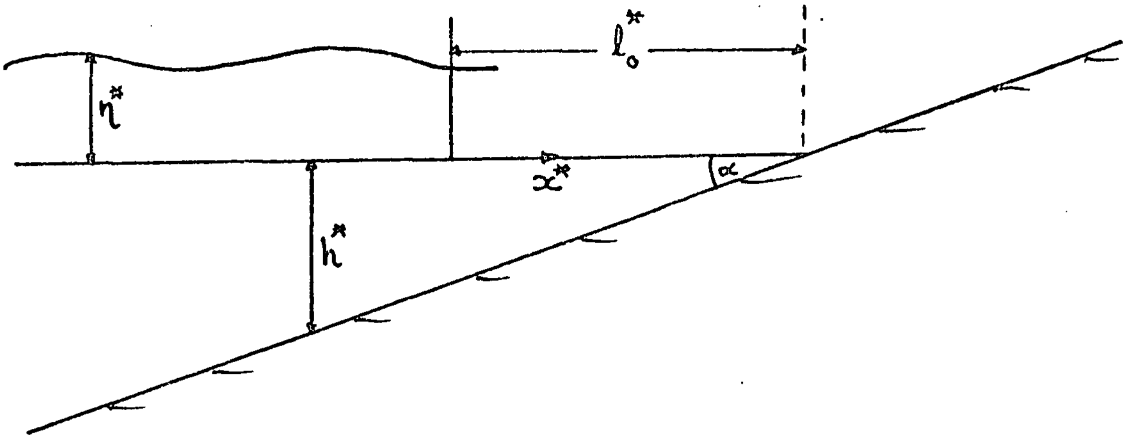


Figure 7.10 Fluid with a fixed boundary and a free surface. (11)

In this paper the characteristic length l_o^* is chosen as the distance of the origin of co-ordinates system from the shore line. When non-dimensionalised this distance became equal to unity.

Since the wave propagation is into water at rest, $v = 0$ in $0 \leq x \leq 1$. Hence the slope of the characteristic at any point (x,t) is given by

$$\frac{dx}{dt} = C \tag{7.16}$$

along which v + 2c + t is constant, and since C = 1 at x = 0, v + 2c + t = 2c + t = 2. That is,

$$\frac{dx}{dt} = 1 - \frac{t}{2},$$

and on integrating, we have

$$x = t - \frac{t^2}{4} \text{ for } t \leq 2.$$

At the wave front the wave velocity is given by

$$C = (1 - x)^{\frac{1}{2}}.$$

Also, at t = 0, the wave front is at the origin of the co-ordinate system fixed in the fluid. In time t the wave front moves through a distance $\int c\,dt$ and if ξ denotes the distance from the moving wave front, then

$$x = \xi + t - \frac{t^2}{4}$$

The new co-ordinate ξ is the position of the wave front and using the basic equations, in terms of the new co-ordinate ξ, the following results are obtained for $u_\xi(o,t)$ and $\eta_\xi(o,t)$,

$$\left(1-\frac{t}{2}\right)u_{\xi t} = \frac{5}{4}u_\xi - \frac{3}{2}\left(1-\frac{t}{2}\right)u_\xi^2 \text{ at } \xi=0, \tag{7.17}$$

and

$$\left(1-\frac{t}{2}\right)\eta_{\xi t} = \frac{3}{4}\eta_\xi - \frac{1}{2}\eta_\xi^2 \tag{7.18}$$

In this case the wave is steadily decreasing at the wave front if $\eta_\xi(0,0) < 0$ and from (7.18) we see that $\eta_{\xi t}(0,0) < 0$. This shows us that the wave front steepens. Also we see that if $\eta_\xi(0,0) = 0$ then $\eta_{\xi t}(0,0) = 0$, showing us clearly that this type of wave cannot break or form a bore at the wave front. Solving (7.17) and (7.18) Greenspan obtained,

$$u_\xi(0,t) = 1/\left[(2-t)\left\{1 - A^{1/2}\left(1-\frac{t}{2}\right)^{3/2}\right\}\right]$$

and

$$\eta_\xi(0,t) = 1/\left[2\left\{1 - A^{1/2}\left(1-\frac{t}{2}\right)^{3/2}\right\}\right]$$

where,

$$A = \left[\frac{u_\xi(0,0) - \frac{1}{2}}{u_\xi(0,0)}\right]^2 = \left[\frac{\eta_\xi(0,0) - \frac{1}{2}}{\eta_\xi(0,0)}\right]^2$$

This shows us that if $\eta_\xi(0,0) < 0$, then A > 1, and the wave breaks when the slope at the wave front becomes infinite, that is, when

$$t = 2(1 - A^{-1/3}) < 2$$

or

$$x < t - \frac{t^2}{2} < 1$$

Therefore, we can say that a steadily decreasing wave (at the wave front) propagating towards the shore with a discontinuity in the surface slope breaks before reaching the shore line.

In both the above papers special forms of shallow water equations are derived and the slope of the bottom is assumed to be constant. Also in both papers the shape of the wave is determined first and hence the conditions for breaking depends on the shape of the wave.

Now we shall look briefly into a method given by Jeffrey (12) whose method of approach could be applied to problems involving sloping beaches having non-uniform slopes. This method employs the Lipschitz continuity property in the neighbourhood of the wave front and does not depend on any special properties of the basic equations. We also find, as expected, that Stoker's (24) results by analytic solution for a special case of waves in water of uniform depth, and Jeffrey's result for zero slope are the same.

In all the papers discussed previously it was shown that the breaking of the wave depends on the initial shape of the wave front. Jeffrey (13) considered the propagation of a smooth fronted wave using shallow water theory, where a smooth fronted wave is taken as one in which the surface slope is continuous across some line in the free surface, but the second derivative of the surface slope is discontinuous across the same line. Using the same techniques as in (12) the author established that in the context of the shallow water wave approximation, smooth fronted waves propagating into still water above an arbitrarily smooth sea bed profile can never break at the wave front until they reach the shore line, after which their behaviour depends on the subsequent motion of the shore line itself.

The study of the climb of a bore on a beach of uniform slope is interesting because of its close resemblance to non-uniform shock propagation in gas dynamics. This problem was examined by Ho and Meyer (20) whose method of approach is based on the fact that non-trivial solutions of the given differential equations have singularities. The differential equations are of the form

$$\phi_{yy} - \phi_{zz} - \frac{k}{z}\phi_z = 0 \tag{7.19}$$

k being a constant.

Equation (7.19) has a singularity at $z = 0$ and the immediate problem here is to specify the necessary and sufficient conditions on the singular line $z = 0$ for the existence, uniqueness and stability of the solutions.

The bore relations given in Section 2 of (20) furnish some of the boundary conditions. The other boundary conditions are given in the subsequent Sections 3 and 4 and they are the seaward boundary conditions

$$\frac{u_b}{V} = 1 - \frac{h_o}{h_b}, \tag{7.20}$$

$$2V^2 = gh_b\left(1 + \frac{h_b}{h_o}\right), \tag{7.21}$$

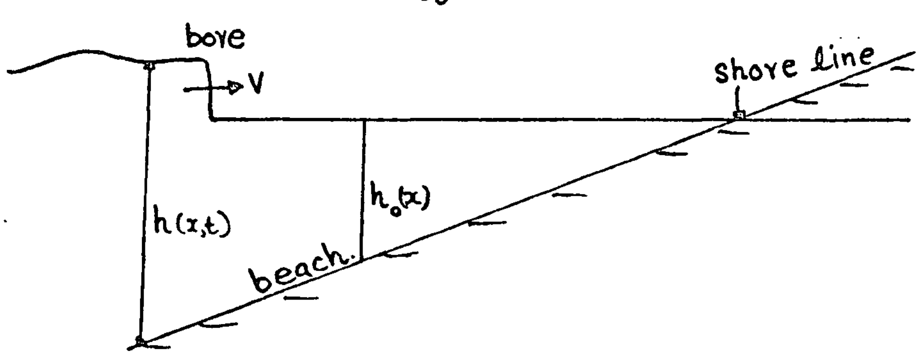


Figure 7.11 (20)

where is the horizontal water velocity, $h_o(x)$, the undisturbed water depth, $h(x,t)$ the total water depth and V the velocity of the bore. The remaining boundary conditions giving specific information about the water motion behind the bore is given in Section 3 of the paper. The bore height $(h_b - h_o)$ is assumed to be a single valued, continuous function of time t, for $t < 0$. Hence $V(t)$, $u_b(t)$ and $C_b(t)$ all become single valued and continuous. Also in Section 3 the authors have defined a limiting characteristic L as shown in Fig. 7.12 whose

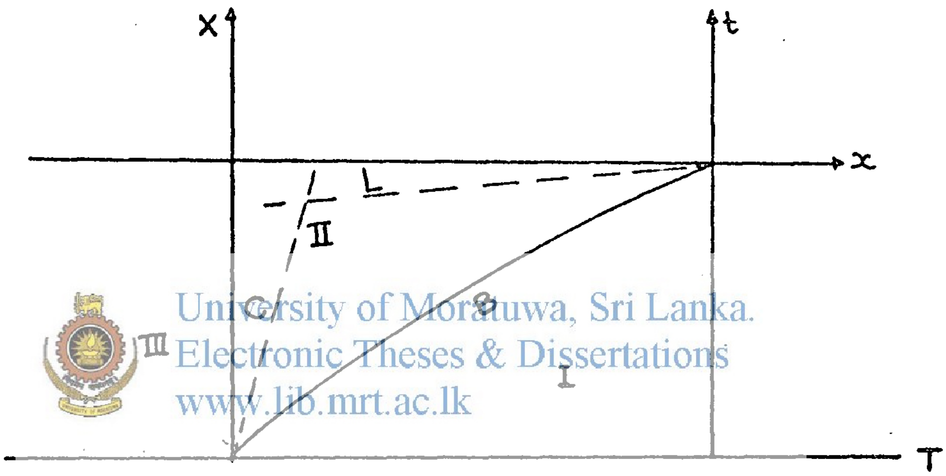


Figure 7.12 Diagram of (x,t) plane showing of successive bore positions. (20)

importance was first pointed out by Guderly. The bore path is given by B in the (x,t) plane. The time is measured from the instant at which the bore reaches the shore. The authors have assumed the bore to be known for $t = T < 0$. C is assumed to be a receding characteristic line of the water motion behind the bore issuing from the bore at time T. So, if u and h are known on the segment of C between B and L, the bore development is uniquely determined in $T < t \leqslant 0$. Mathematically, this segment of C is called the seaward boundary. A fairly detailed qualitative approximation for the solution near the shore is obtained in Section 5 of the paper and the approximate bore path is also obtained.

Meyer and Shen (21) proceed to discuss the climb of a bore on a beach having a non-uniform slope. Here too it is shown that the shore singularity for a beach of uniform slope still gives an approximate solution and that the shape of the beach affects only the basic velocity of the bore in the development of the bore close to the shore. It is also shown in this paper that the main results regarding the shore singularity are not exceptional as in the first part of the paper. It is also assumed that the bore reaches the shore at a finite time as in (20) which is taken as t = 0. With these assumptions the authors obtained the following relations

$$c_b \to 0 \quad \text{as } t \to 0$$
$$h_b > h_o \quad \text{for } t < 0$$

$$u_b + c_b > v_b > c_b > 0, \quad v_b > u_b > 0 \quad \text{for } t < 0$$

They also showed the existence of a limiting characteristic L as in Figure 7.12. By assuming a shorter time interval during which the bore development is studied, the existence of any secondary bores is eliminated in the region II bounded by B, L and C in Figure 7.12.

In Section 3 of the paper, the authors established the existence of an asymptotic approximation which depends on the velocity u_b as well as on the beach slope variations. For the bore condition $\frac{dx_b}{dt_b} = v_b$ an equivalent expression is obtained as

$$(u - v_b - c)t_b \frac{d\alpha_b}{d\beta} + (u - v_b + c)t_\beta = -\left(1 + \frac{d\alpha_b}{d\beta}\right)\delta j$$
$$\text{on} \quad \alpha = \alpha_b(\beta)$$

The other bore conditions as in (20) are also satisfied asymptotically since $h_o(x)$ is continuous. This shows that asymptotic solutions in (20) satisfy the shallow water equations and bore conditions asymptotically to a first approximation on a beach of non-uniform slope.

Tsunami Waves

The results of many recent observations suggest that tsunami waves consist of a train of several large, approximately sinusoidal waves of about 1 m in height moving in the deep ocean at approximately the shallow water speed of $\sqrt{gh}$. Mader (19) studied this using the Marker and cell method which is a technique for the calculation of viscous, incompressible flow with a free surface. This method uses a finite difference technique for solving the time dependent Navier-Stokes equation.

In the shallow water theory we assumed that the vertical component of the motion does not influence the pressure distribution, which was assumed to be hydrostatic. The results obtained by using the two methods were compared in several cases by Mader.

Mader used single solitary like waves to demonstrate the fundamental features of the flow and for checking the numerical results obtained from tsunami waves. The results obtained for several models are illustrated in the paper (19). The models are, (a) one metre half-height; 1320 seconds tsunami, Fig. 7.13; (b) one metre half-height, 660-seconds tsunami, Fig. 7.14;(c) half metre half-height, 660-seconds tsunami, Fig. 7.15. In all these cases the author gives a comparison of solutions obtained using the SWAN and ZUNI codes (see the paper by Mader).

Further, the author discusses some underwater barrier results. A submerged barrier absorbs some of the wave energy and the wave consequently breaks prematurely. Also much of the wave energy is reflected back seawards.

However, tsunami waves are of sufficiently long wave length that they do not tend to break. Thus underwater barriers will be effective on tsunami waves only as reflectors of energy. This implies that the shallow water theory is inadequate to determine the effect of underwater barriers on tsunami waves because of the importance of the vertical velocity on the flow.

Many calculations have been made assuming the location of barriers under water. Mader's work shows us that the numerical simulation of gravity waves resembles the profile of actual tsunami waves. The wave heights were observed to increase by a factor of 4 as they shoaled up a 1:15 continental slope and the results obtained by using the shallow water theory for long wave length tsunamis were similar. But for short wavelength tsunamis it was different.

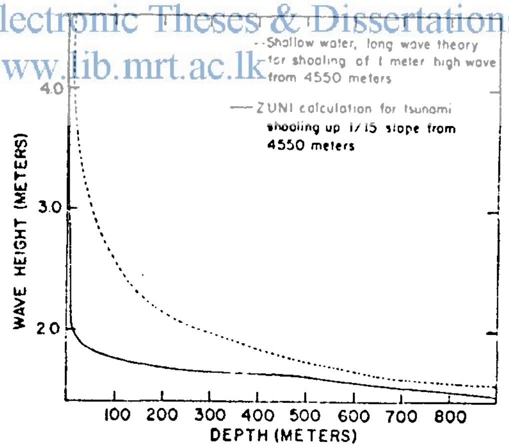


Figure 7.13 The amplitude of calculated 1 m half-height, 1320-sec. tsunami waves as they shoal up a 1:15 slope from 4550 m. Also shown is the shallow water, long-wave curve. (20)

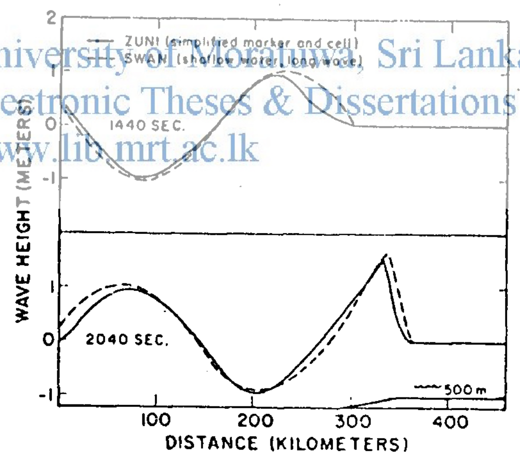


Figure 7.14 Computed wave surface profiles for a 1 m half-height, 1320-sec tsunami interacting with a 1:15 continental slope, a continental shelf 500 m deep and reflecting off a cliff; and the shallow-water, long-wave calculations for the same model. (20)

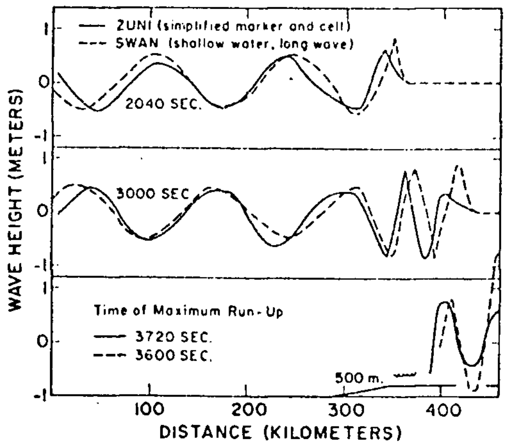


Figure 7.15 Computed wave surface profiles for a 0.5 m half-height 660-sec tsunami interacting with a 1:15 continental slope, a continental shelf 500 m deep and reflecting off a cliff, and the shallow water, long wave calculations for the same model. (20)

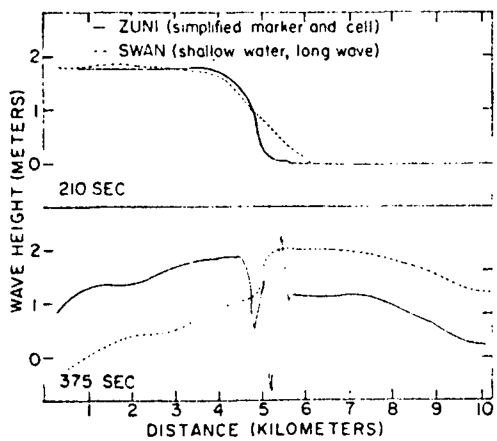


Figure 7.16 Surface wave profiles for a 660-sec tsunami interacting with a 11.1 m deep barrier, and the shallow water, long-wave profiles for the same model. (20)

8. General remarks

In all the papers we have mentioned the authors have based their discussions on the shallow water wave theory by Stoker, but with modifications depending on the problem such as 'the breaking and climbing' of a wave depends on the type of wave and the bottom topography of the sea bed.

The shallow water wave theory itself is an approximation and the results obtained will be accurate only to a certain degree. Thus any further modifications or approximations will tend to make the results less accurate.

Therefore we can say that there is still no general criteria found which enables us to determine when a given wave will break other than at the wave front, although the initial wave shape and the bottom topography are of fundamental importance.

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